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1 J.L.SYNGE

on

2 WHITEHEAD'S PRINCIPLE OF RELATIVITY.

2.1 APPENDIX A: Solar Limb-Effect; B: Figures

2.2 Critically edited by A. John Coleman,

2.3 Queen's University, Kingston, ON, Canada.

2.4

2.5 Dogmatic Opinions and Objective Thoughts of the Editor. *colemana@post.queensu.ca*

It was the opinion of Professor Synge in 1951 - probably until his death -, as it is my opinion in 2005 that the evidence for the validity of Einstein's and of Whitehead's theories of Gravitation is roughly of equal value. Neither Synge nor I would claim that either is "Correct" . Certainly, Whitehead would be the last to do so. I refer to these theories as GRT and PR.

Since GRT is the dominant faith among current relativists, am I not, in the words of Synge's Introduction, "attempting to exhume a corpse" by mentioning Einstein's and Whitehead's Theories in the same breath? Has not our Sacred College[1] spoken, dismissing Whitehead with faint praise and a passsing reference, in 1970?

Both PR and GRT presuppose the validity of Einstein's Special theory(SRT). C.M. Will, in his discussion[2] of PR, states in foot-note (3) that, as regards theAdvance of the perihelion, the Bending of light and the Retardation of electromagnetic signals, the two theories are both within the limits of observation. However, he also claimed , by a complicated argument concerning the local gravitational constant, G, that he had administered the *coup de grace* to PR just as, in my opinion, *QM has done to GRT.*

2.5.1 (i) John Lighton Synge (1897-1995)

Synge wrote 13 books and over 200 papers exemplifying the lucidity and sense of drama that all of us who had the privilege of attending his lectures knew and admired. Most notable perhaps were his treatises[3] on Relativity: SRT in 1956, and GRT in 1960. Although these are widely regarded as the logically clearest available presentations of Einstein's Theory, a careful reading of his text reveals that he explicitly refuses to say that he considers GRT to be "correct".

Indeed, in essence, on pp. IX/X of the Preface of his volume on GRT, he says that while the Principle of Equivalence played a useful role in firing imagination it makes no sense and that by 1960 should have been abandoned!. It was because Whitehead explicitly rejects this "Principle" that in 1922 he was able to predict the solar limb-effect noted by Halm in 1907.

In my senior year, 1938/9, as an undergraduaate at the University of Toronto I had the good fortune to audit Synge's Graduate Course on GRT. At one point, I asked "Sir, what do you make of Whitehead's criticism of Einstein's theory of measurement?" . He brushed me aside curtly, saying "I never heard of it!" and went on, leaving me in embarrassed silence. By 1950 , Synge and his former co-author, Alfred Schild, were the centre of a minor revival of interest in PR. Later, in a Public Lecture in Toronto he stated that he had become disenchanted with GRT, my question came to his mind and he decided to look seriously at W.'s theory. As far as I am aware he was the first conpetent Mathematical Physicist to do so except, possibly, Eddington.

In 1952 he invited me to give eight lectures to his seminar at theDublin Advanced Institute to a small group which included Schroedinger - a challenging task for a 34-year Assistant Professor! These dealt with my thesis, boldy entitled "Relativistic Quantum Mechanics" , which was based on the late work of Eddington about the Comstants of Nature.

Although I became Synge's "walking companion" and, later, his friend , unfortunately I never discussed Whitehead with him since I did not learn of his Maryland Lectures until after his death.

I feel that Synge's Lectures are of great importance in the current discussion about Relativity, Quantum Mechanics and the Theory of Everything. I decided to make them more easily available.

2.5.2 *(ii)* The Lectures

What follows is a Portable Latex version of a hard copy of the original duplicated notes of Synge's three Lectures given at the University of Maryland in 1951. Mathematical symbols had been entered by hand so occasionally were difficult to read. There was one serious error which made two pages unintelligible. I hope, but cannot guarantee, that my revision is correct.

In three lectures it was impossible for Synge to do justice to the subtlety of thought about space, time and matter set forth in Whitehead's writings on physics, beginning in 1905, with a profound essay for the Aristotelian Society, to 1922, culminating in his three books[4], The Principles of Natural Knowledge(PNK), The Concept of Nature(CN), and The Principle of Relativity(PR).

Synge gives a clear elegant summary , in modern tensor notation, of the mathematical arguments in seven of the 13 chapters of Part II of PR . But he makes no allusion to the 4 Chapters of Part I which attempt to explain what scientists in general and astronomers in particular mean by terms such as "metre" or "second". Nor did Synge have time to discuss Part III in which Whitehead explains his use of Tensors. For me as an undergraduate , Part III was a revelation of the real significance of tensors and their relation to the theory of Group Representations.

The chapters which Synge chose to discuss provide the immediate material needed to compare the implications of PR for the tests which were advanced for GRT and for which PR and GR are equal but he makes only a brief allusion to PR, Ch. XIV on the Limb Effect.

Synge's discussion is correct though condensed. However, I regard it as a mistake of exposition that he replaced ANW's notation, dJ^2 for the"gravitational potential" by ds_g^2. It could encourage the casual reader to miss tANW's main criticism of GRT stated on p. 83 of PR, to which I return in in (*iv*) below.

I am grateful to Cathleen Synge Morawetz - Professor at the Courant Institute, former President of the American Mathematical Society, Synge's only child and his Literary Executor - for encouraging me to complete this project in order to spread knowledge of a little-appreciated aspect of her father's work.

I am also greatly indebted to Jordan Bell - a sophomore in the School of Mathematics of Carleton University, Ottawa - for turning an imperfect Mss. into Latex with admirable dispatch and remarkable insight.

2.5.3 (iii) Alfred North Whitehead (1860 -1947)

2.5.4

ANW, as I shall refer to him , is now chiefly renowned as a philosopher whose famous book[5] - subtitled "An Essay in Cosmology" was the focus of the new movement known as *Process Philosophy.* This occurred after 1923 when he became a Professor at Harvard. From 1884 to 1910, his position at Trinity College, Cambridge required him to prepare students for the Tripos in the whole range of Pure and Applied Mathematics. Even so, in that period he created a new branch of mathematics with his *Universal Algebra*[6]. Then with his student , Bertrand Russell, wrote the three volume *Principia Mathematica.* Thoughout this extraordinary creative period however, it seems that his favourite subject for lectures was Maxwell's Theory! This confirms my interpretation of Whitehaed as, at heart, a mathematical physicist attempting to understand the Universe in all its aspects.

2.5.5 (iv) WILL'S "coup de grace"

I have not succeded in understanding Will's argument[2] so, while I have strong reservations about its validity, I cannot say that it is incorrect. The paper was put together during a period when Will was finishing Ph. D. studies. He mentioned that PR is a difficult theory which is not in the category of the "metric" theories to which the *parameterized post-Newtonian(*PPN) notation is applicable. Yet it is PPN that he uses to dismiss PR. I first became suspicious of Will's argument on reading his Foot- Note(10) which ascribes to Whitehead the opposite of my understanding of Whitehead's clear assertions.

The only responsible critical discussion of Will's paper[2] of which I am aware is that of D.R.Fowler [7] who claimed that it contains both physical and philosophical errors.

(*a*) Fowler points out that Will assumes in his critique of PR that the mass of our Galaxy is concentrated at the centre of the Galaxy whereas, by smearing the mass uniformily, Will's estimate of the error predicted by PR is reduced by a factor of 100! The force of Fowler's argument is enhanced by the presence of Dark Matter which was unknown when Fowler or Will wrote.

. (*b*) Fowler states further that Synge and, following him, Will have quite failed to understand the real meaning of Whitehead's theories. Whitehead remarked in PR that it would be helpful to read the two previous books[4] in which he developed his concept of space and time and the role of Lorentz Transformations. Indeed, I found this essential in the sporadic sallies that I took since 1937, to penetrate the thought of Whitehead re. Physics. Yet I found no reference to CN or PNK in the writings of Synge, Will or MTW!

If Fowler is correct it would imply that PR was banished from serious conisderation by mainstream Physics because of a paper which estimated its discrepancy as 100 times the actual amount. This suggests the desirability of a careful review of rhe papers of Will and of Fowler by an independent critical study.

Whitehead stated that Einstein's theory of measurement involves a basic inconsistency: one does not know the meaning of "distance" between two events, specified initially by physically meaningless co-ordinates, until Einstein's Equations have been solved with initial conditions given in terms of "metre" and "second" which cannot be defined until the equations have been solved! I have never seen this criticism directly addressed in the Literature of GRT. Possibly MTW thought they did so with their cute story of the student, the ant and the apple in the opening pages of their famous treatise. But what this story implies is that you can give a meaning to "distance" if you are an ant (or astronomer) who has solved the equation for geodesics - which you are unable to formulate.

Only if you are born with a God-given intuitive ability to follow geodesics in space-time would you be able to measure anything!

This is such a clear simple inconsistency that I see no way to avoid it.

2.5.6 (*v*) LIMB-EFFECT

The average of frequencies of a line in the solar spectrum at extreme ends of the equator should remove the Doppler Shift. One expects this average to equal the frequency observed at the centre of the Disk for the same line. The first observation that this is seldom the case, now named the "limb-effect", is attributed to Halm in 1907. I have been unable to discover anything about Halm and would appreciate enlightenment!

IF one accepts GRT including the Strong Equivalence Principle($SSEP$) , it follows that spectral shifts due to gravity are proportional to the frequency of the line . Thus GRT can throw no light on the limb-effect.

Synge, like Whitehead, does not accept the Equivalence Principle.

"Whitehead's theory of relativity implies that there is an interaction between the gravitational and electromagnetic fields such that for an atom at the surface of a star, the Coulomb potential r^{-1}between two charges must be replaced by

$$\frac{1}{r}(1 - \alpha cos^2\theta). \tag{(1)}$$

Here, θ is the angle between the radius vector joining the two interacting charges and the direction of the stellar radius passing through the *a*tom; α is a small constant depending on the strength of the gravitational field. At the surface of the sun, $\alpha = 2.12$x 10^{-6} approximately.

The effect of (1) is to perturb the normal energy levels by the term

$$-\alpha\frac{\cos^2\theta}{r} \tag{(2)}$$

which has axial symmetry about the stellar radius through the atom. An effect of precisely this symmetry is what is needed to explain the limb-effect in the solar spectrum. One might also hope that this perturbation could account for the striking differences which have been observed in shifts within the same solar multiplet."

This long quotation is fron my 1968 paper reproduced in the Appendix to which the reader is referred.

I am not aware of any serious attempt to explore the consequences of PR in this connection. Yet a huge effort at great expense is devoted to theories about the solar atmosphere which are based on the GRT formula.

It has frequently been remarked that Solar Spectral shifts depend not only on Doppler and pressure effects but also on the energy levels of the terms of the transition of which the line gives evidence. Miss Adam[8] noted that shifts within one multiplet frequently differ by amounts of the order of the basic prediction of Einstein's formula.

I therefore propose, as a first step in unravelling the apparent complex confusion in observed Solar spectral shifts, that an extensive precision study be made of shifts within single Multiplets and that an effort be made to obtain a theoretical understanding of them, using Whitehead's formula above or otherwise. It is conceivable but highly unlikely that the results of Adam could be explained as a pressure effect so their verification might well sound the death knell for SEP and even for GRT in its present form. Certainly, such would be the case if they are explicable by the above formula of Whitehead.

This issue is highly significant since it bears not only on our speculations re motions in the solar atmosphere but also on all the cosmological theories for which spectral shifts are frequently the ONLY available relevant data. It may well happen that the α, in Whitehead's formula (2), is quite important for small dense stars.

2.5.7 *(vi)* Re*lativity* and Quantum Mechanics

It is widely, perhaps even universally , recognized that the most important unsolved problem in physics is how to reconile these two great theories or, at least to harmonize them. In my (dogmatic) opinion no significant progress has occurred since the work of Eddington in the mid 1930's His ideas were dismissed as "speculative" or "philosophical". In those days ,this latter word was the kiss of death! In fact, the ideas of Eddington were dull traditional physics , compared with the wild speculations and wonderful TV to which String and M-theory have given rise.

Eddington had a very simple idea. Assume, as he did , that GRT and QM are both correct, Choose a pproblem which can be solved by both. He chose a finite uniform universe. Equate the two solutions leading to a relation between ■ and λ This was the basic step which led him to *calculate* seven Constants of Nature accurately to 1 in 10^4. Perhaps the stongest observational support any Theory ever had. It was rejected immediately because it was too "philsophical" but the logical conclusion was not noticed: *one or both GRT and QM is/are invalid!*

In the preface, to the second edition(1958) of his famous reatise on GRT , W. Pauli, one of tthe principal creators of QM, wrote

" .... a clear connexion between the general theory of relativity and quantum mechanics is not in sight." .

In my opinion, in the past three decade the hope of finding such a connection has disappered in a murky fog created by the many conflicting claims ,and coounter-claims and wild speculation of proponents of Gauge, String and M theories. An attractive counter opinion to mine can be found in the 1998

Cambridge treatise: STRING THEORY , by Joseph Polchinski or in MATHEMATICAL PHYSICS, 2000, Imperial College Press, the preparatory volume for the International Mathematical Congess at Imperial College, in 2000.

I came awayfrom the Comgress , with the feeling that the ablest physics Graduate Students were being led into wasting their minds on a misdirected course chasing a chimera. Over beer, two very competent Cambridge Graduate students bemoaned the fact that they had gone to Cambridge hoping to learn some exciting Physics but were struggling with exotic mathematics with no connection to Physics.

I taught GRT enthusiastically for eight years but am now disenchanted. It is an all-or-nothing Theory. Whitehead is more modest. He proposed PR as a first step beyond Newtonian Physics taking acccount of SRT. This is what QM needs and may well be comfortable in PR. There is need for a follow-up of Rayner's paper exploring the cosmological implications of PR - (cf. Rayner,1954, Proc. Roy. Soc., London A **222, 509;** and Synge, Proc.Roy. Soc., London **A 226**, 336)J. I assume that such has not been pursued hitherto because of the common obsession with GRT.

NOTES

1. MTW: C.W. Misner, K.S. Thorne, J.A. Wheeler authors of GRAVITATION, W.H. Freeman &Cpy, 1970.
2. Clifford M. Will, Astrophysical Journal, **169,** 141-155**,** (1971)
3. (*a*) RELATIVITY -The Special Theory, 1956 (*b*) RELATIVITY -The General Theory, 1960 The North Holland Publishing Cpy.
4. PNK: An Enquiry Concerning Natural Knowledge. CUP, 1919.

CN: The Concept of Nature, CUP, 1920
PR: The Principle of Relativity with Applications, CUP, 1922; Reprinted, Dover, NY, 2004.

5 Process and Reality, MacMillan, 1929

6 A treatise on Universal Algebra with Applications,Vol. 1 , Cambridge University Press, 1898. Apparently, ANW intended to write another volume devoted to geometry but this plan was interrupted by working with Bertrand Russell on the Three volumes of Principia Mathematica.

7 Dean R. Fowler, in PROCESS STUDIES, **4,** 4 (1974). For an insightful discussion of the central ideas of Whitehead's approach to physics, see also Fowler's article in ibid **5,** 159-174 (1975).

8 Madge Gertude Adam(1912−2001) was a Solar Astronomer at the Oxford Observatory from 1935 until her death at 89. Her work on the nature and magnetic fields of Sunspots gained international recognition. However, she also took great interest in the Limb Effect to which two of her widely quoted papers were devoted: Mon. Notices RAS, **119**, 460-470(1959) ; ibid. **177**, 687-707(1976) . In these and several other papers she reports shifts for various multiplets.

These I found particularly interesting since one would expect that, if the effect were due to a perturbation proportional to the line frequency, shifts of different lines of a multiplet would be equal . This is not observed. I therefore visited the Oxford Observatory to discuss these observations with Miss Adam. She was paarticularly keen on the observations for iron lines since she felt that for these her results were quite reliable.

This conversation together with the fact, already noted by St John in the twenties, that the shift depends on the excitation potential associated with the line, redoubled my conviction that observed shifts within a ingle multiplet could be the key to understanding the solar spectrum.

1. A. John Coleman, Feb.7, 2005

3 Synge's Lectures

3.1 Lecture I: BASIC HYPOTHESES

3.1.1

3.1.2 1.1 **Introduction**

In 1922 there appeared a book by the late Professor Alfred North Whitehead entitled *The Principle of Relativity* (Cambridge University Press). His book contains a clearly formulated theory of gravitation and of electromagnetism in a gravitational field, and so invites comparison with Einstein's General Theory of Relativity which had appeared six or seven years earlier.

In attempting such a comparison, one becomes aware of certain psychological factors. The philosophy of science, being on the whole very little discussed among active physicists, is naive in the sense that many things are taken for granted subconsciously. One may believe, for example, that the laws of nature, including the ones of which we today know nothing, lie locked in blueprints in a filing cabinet, and that the achievement of the human mind in theoretical physics is less an act of creation than a successful burglary performed on this filing cabinet.

If that view is accepted, then once the real blueprint has been exhibted to the world, any further blueprints for the same structure brought forward at a later date must be dismissed as forgeries. There is no room for two different theories covering the same phenomena; if one is right, then the other must be wrong. That, it may be claimed, is a fair statement of a view widely, if silently, held. It is a view very difficult to dismiss from our minds, trained as they are in an old and little-discussed tradition.

Anyone who attempts to describe Whitehead's theory of gravitation has to face this attitutde on the part of physicists, and if he is a physicist himself his own thughts are not immune from it. He cannot help feeling that he may be swimming against the tide, or (to change the metaphor) exhuming a mummy instead of trying to contribute to the growth of the living body of physics. To bolster his own confidence, he may be tempted to create an artificial enthusiasm as a device of propaganda in order to win a hearing for a theory which has slipped away into a fairly complete oblivion.

There is another difficulty in describing Whitehead's theory. Whitehead was a philosopher first and a mathematical physicist second. How can one who is not a philosopher attempt to describe the work of a philosopher? Certainly he cannot, if the work of the philosopher is philosophy. But if the philosophy is only a wrapping for physical theory, then the mathematical physicist can take a savage joy in tearing off this wrapping and showing the hard kernel of physical theory concealed in it. Indeed there can be little doubt that the oblivion in which this work of Whitehead lies is due in no small measure to the effectiveness as insulation of what a physicist can in his ignorance describe only as the jargon of philosophy. The account of Whitehead's theory given in these lectures is emphatically one in which the philosophy is discarded and attention is directed to the essential formulae. And this, it may be claimed, is as it ought to be. No student of Newtonian mechanics should be asked to reconstruct it in the form in which it appeared to Newton himself.

The practical physicist who lives among the facts of observation is naturally impatient of theories except in so far as they assist him in understanding those facts. He is entitled to ask, of Einstein's theory of gravitation and of Whitehead's, whether they adequately explain the facts.

In the case of Einstein's theory, the answer might be put like this: For slowly moving bodies and weak gravitational fields, the Einstein equations yield a close approximation to Newtonian gravitation. Since the velocities of celestial bodies are for the most part slow (in comparison with the velocity of light) and the gravitational fields are weak, we are to expect only minute deviations from what is predicted by Newtonian mechanics. Three small deviations in the solar field are predicted and verified by observation within the limits of the errors of observation; these are rotation of the perihelion of Mercury, deviation of a light ray passing close to the sun, and spectral shift towards the red in a gravitational field.

In the case of Whitehead's theory, all the above statements hold. By what appears to be a rather extraordinary coincidence, there is a formal agreement between the two theories in the matter of particle orbits and light rays, and we find ourselves in the strange position of having two theories which both appear adequate on the basis of observation.

Such a situation is rather unusual in physics. The facts of observation are so numerous that it would appear easy to find the crucial observation which would decide in favor of the one theory or the other, or perhaps controvert them both. The difficulty here lies in the close agreement of both theories with Newtonian mechanics, with the result that the critical differences are necessarily very small.

There is however a difference in the character of the two theories. Einstein's theory is based on a set of non-linear partial differential equations involving a somewhat ill-defined term, the energy tensor, and specific applications are exceedingly difficult to work out. The difficulties are not merely those of mathematical manipulation. There are deeper difficulties in the sense that one is hardly convinced that certain problem are clearly formulated; thus in spite of the ingenious manipulations used in handling the n-body problem,[1] the question remains as to whether these "bodies" are singularities in the field, and, if so, what is meant by a singularity in a Riemannian space with indefinite line-element.

In respect of clarity, Whitehead's theory has much to recommend it, because it is not a field theory in the technical sense. The problem of n-bodies can be unequivocally formulated, and the difficulties of solving it are purely mathematical. Or to take a terrestrial example, it is possible to formulate mathematically the problem of a fast-moving particle in the presence of heavy fixed masses on the earth's surface, a relativistic refinement of the problem of geodetic observations. In fact Whitehead's theory of gravitation has an applicabilitiy which the General Theory of Relativity lacks by virtue of the fact that the latter is a non-linear field theory.

This would appear an opportune occasion to venture an expression of disagreement with what may be called the "Filing-Cabinet Theory of Scientific Theories" (as discussed earlier) and to suggest that scientific theories might be viewed as statues or models of which there may be several representing one thing, but definitely man-made and subject to rejection, destruction, modification and cannibalisation.

What is given in these lectures is a very free translation of Whitehead's theory. As indicated above, there is no attempt at all to reconstruct Whitehead's philosophy. I have not thought it necessary to keep his notation. But I hope that I have not tampered with the essentials of the theory, namely, the choice of the axiomatic formulae from which everything else follows.

[1] Einstein, Infeld and Hoffman, Annals of Mathematics 39, 65 (1938); Einstein and Infeld, Annals of Mathematics 41, 455 (1940) and Canadian Journal of Mathematics 1, 209 (1949).

3.1.3 1.2 The Minkowskian Background

The first essential thing to observe about Whitehead's theory is that it uses the space-time of the *Special Theory of Relativit*y, or, more correctly, the space-time of Minkowski. Mathematically, this means that we consider a four-dimensional contiuum of events and in it certain privileged systems of coordinates (x,y,z,t) related to one another by Lorentz transformations. Given any two adjacent events, their "separation" has a value

$$dx^2 + dy^2 + dz^2 - c^2 dt^2$$

independent of the particular system of coordinates used. These special coordinates we may call Galileian. We shall not consider the generalisation of Whitehead's theory to curved space-time.

The constant c occuring above is a universal constant with a value depending on the chosen units of space and time. We do not say here that c is the velocity of light, since light is to be regarded as an electromagnetic phenomenon, and as such will be discussed later.

In this Minkowskian space-time the history of a particle is a curve, as we are accustomed to think of it in the Special Theory of Relativity. At this point we come to the essential hypotheses of the Whitehead theory, which may be presented as answers to the following two questions:

(a.) What is the gravitational field due to a particle?

(b.) How does a particle move in the gravitational field due to itself and to other particles?

3.1.4 1.3 The field due to a particle or to a set of particles and the equations of motion

Let us use the following notation:

$$\text{space coordinates } \; x_\alpha (\alpha = 1, 2, 3) \text{ for } x, y, z$$
$$\text{time coordinate } \; x_4 = ict.$$

Throughout we shall give Greek suffixes the range 1,2,3 and Latin suffixes the range 1,2,3,4, with summation on repetition in both cases. Thus our space-time coordinates are x_r and the fundamental form, invariant under Lorentz transformation, is

$$dx_r dx_r. \tag{3.1}$$

Except for special use of curvilinear coordinates on occasion, tensorial properties are with respect to Lorentz transformations only. In terms of the Minkowskian coordinates x_r, there is no distinction between covariant and contravariant tensors; we shall in general use subscripts to indicate the components, rather than superscripts.

For any element dx_r we write

$$ds^2 = \epsilon dx_r dx_r, \tag{3.2}$$

$\epsilon = 1$ for space-like directions,

$\epsilon = -1$ for time-like directions,

the sign being chosen to make ds^2 positive. For a time-like element, ds is the measure of proper time. The unit tangent vector to a world line is

$$\lambda_n = dx_n/ds. \tag{3.3}$$

Consider now a time-like world line L' (Fig. I.1) and a point-event P which does not lie on it. From P draw the null cone into the past, cutting L' at P', say. Let x_n be the coordinates of P and x'_n those of P'. Write for brevity

$$\xi_n = x_n - x'_n, \tag{3.4}$$

so that from the null property,

$$\xi_n \xi_n = 0. \tag{3.5}$$

Draw the tangent to L' at P' and drop the perpendicular PN on it. Then, in an obvious notation, for vectors,

$$NP_n = \xi_n - |P'N|\lambda'_n \tag{3.6}$$

where λ'_n is the unit tangent vector to L' at P'. From the orthogonality at N we have

$$NP_n\lambda'_n = 0, \tag{3.7}$$

and so, since $\lambda'_n\lambda'_n = -1$,

$$|P'N| = -\xi_n\lambda'_n. \tag{3.8}$$

Then, from (3.6), since NP_n is a space-like vector,

$$|NP|^2 = NP_nNP_n = [\xi_n + (\xi_p\lambda'_p)\lambda'_n] \tag{3.9}$$

$$\times[\xi_n + (\xi_q\lambda'_q)\lambda'_n] = (\xi_n\lambda'_n)^2. \tag{1}$$

In view of the sign in (3.8), we have then

$$|NP| = w, \tag{3.10}$$

where

$$w = -\xi_n\lambda'_n. \tag{3.11}$$

If L' is straight, and if it is taken as the time axis, then $\lambda'_\alpha = 0, \lambda'_4 = i$, and

$$w = -i\xi_4 = c(t - t'). \tag{3.12}$$

Thus if r is the spatial distance of P from L', we have

r = c$(t - t') = w$,

when we use the fact that$P'P$ is a world line corresponding to velocity c. In general, when L' is not straight, the invariant w, as defined in (3.11), plays the role of "distance" of P from L', and it is in terms of w that Whitehead defines the gravitational field due to a particle.

Consider now two adjacent events P and Q, with coordinates $x_n, and \quad x_n + dx_n$, and a time-like world line L', representing the history of a particle (Fig. I.2).

From P and Q draw null cones into the past, intersecting L' at P', Q' respectively, with coordinates $x'_n, \quad$ and $x'_n + dx'_n$. We proceed to find the element of proper time $P'Q'$, which depends only on the events P, Q and the world line L'.

Let the equation of L' be $x'_n = x'_n(s')$, s' being proper time on L'. Then we have

$$(x'_n - x_n)(x'_n - x_n) = 0, \tag{3.14}$$

and this is an equation for the determination of s'. On variation of x_n, we get, in the notation of (3.4),

$$\xi_n(dx'_n - dx_n) = 0. \tag{3.15}$$

But $dx'_n = \lambda'_n ds'$, and so

$$wds' + \xi_n dx_n = 0; \tag{3.16}$$

hence

$$\partial s'/\partial x_n = -\xi_n/w, \tag{3.17}$$

and from this

$$\frac{\partial x'_m}{\partial x_n} = \frac{dx'_m}{ds'}\frac{\partial s'}{\partial x_n} = -\lambda'_m \frac{\xi_n}{w}. \tag{3.18}$$

Then, for the element $P'Q'$ we have

$$\begin{aligned} dx'_r &= -\frac{\partial x'_r}{\partial x_s} dx_s = w^{-1}\lambda'_r \xi_s dx_s, \\ {ds'}^2 &= -dx'_r dx'_r = -w^{-2}\lambda'_r \xi_s dx_s \lambda'_r \lambda_t dx_t = w^{-2}\xi_s \xi_t dx_s dx_t. \end{aligned} \tag{3.19}$$

With these formal preliminaries cleared away, we come to Whitehead's definition of the field due to a particle of mass m and world line L': *the field at P is given by the tensor g_{mn}, where g_{mn} is symmetric and such that*

$$g_{mn} dx_m dx_n = dx_n d_n + (mk/w){ds'}^2, \tag{3.20}$$

for arbitrary dx_r, where ${ds'}^2$ is as in (3.19)and k is a universal constant.

We recognize in mk/w the analogue of the Newtonian potential.

An equivalent expression of (3.20) is

$$\begin{aligned} g_{mn} &= \delta_{mn} + \widetilde{g}_{mn}, \\ \widetilde{g}_{mn} &= (mk/w^3)\xi_m \xi_n, \\ w &= -\xi_n \lambda'_n \end{aligned} \tag{3.21}$$

This "derivation" of (3.20) or (3.21) is in the old tradition of theoretical physics, in which one seeks to lure on the reader step by step from the simple to the complicated. This is interesting historically, because it shows how the final result was built up in the mind of author, but the bald fact is that (3.21) is Whitehead's definition of the field due to a particle, and it must stand or fall, irrespective of the way in which it was built up, on its merits as a predictor of correct observational results.

The field of *one* particle is then set down as in (3.21). If we have a set of particles with $\widetilde{g}^{(1)}_{mn}$masses $m^{(1)}, m^{(2)}, \ldots$ and world lines $L^{(1)}, L^{(2)}, \ldots$, the field due to them all is defined to be

$$
\begin{aligned}
g_{mn} &= \delta_{mn} + \widetilde{g}^{(1)}_{mn} + \widetilde{g}^{(2)}_{mn} + ..., \\
\widetilde{g}^{(p)}_{mn} &= \frac{m^{(p)}k}{w^{(p)3}} \xi^{(p)}_m \xi^{(p)}_n, \\
w^{(p)} &= -\xi^{(p)}_n \lambda^{(p)}_n, \;\; p = 1, 2, 3...
\end{aligned}
\tag{3.22}
$$

the notation being obvious (no summation for p).

Thus, as in Einstein's theory, the gravitational field appears as a symmetric tensor g_{mn}. But there is an important difference, because in Einstein's theory there is no space-time defined topologically which in the g_{mn} are functions of position, and the g_{mn} seem to have then the responsibility of determining the topology. The assumption that there is no measure of separation in space-time except $g_{mn}dx_m dx_n$ leads to some oddities, for if a particle is a singularity at which some of the g_{mn} become infinite, it may become inaccessible through the infinite length of world lines drawn to it. This sort of thing cannot happen in Whitehead's theory, because the gravitational field is displayed against a flat 4-space with Euclidean topology, and an infinity in the g_{mn} causes no embarrassment at all (anymore than does an infinite potential in Newtonian gravitation).

We have now to supplement the hypothesis (3.22) with a further hypothesis regarding the motion of particles. For this Whitehead makes a hypothesis very like that of Einstein: *The world line of a particle satisfies the variational principle*

$$\delta \int d\bar{s}_g = 0, \tag{3.23}$$

where

$$d\bar{s}^2_g = -\bar{g}_{mn} dx_m dx_n, \tag{3.24}$$

$\bar{g}_{mn}$ *being the same as the* g_{mn} *of (3.22) but with the omission of the* $\widetilde{g}_{mn}$ *which is due to the particle itself.* In fact the self-field is omitted, just as in Newtonian mechanics.

3.1.5 1.4 The field of a particle at rest

If a particle exists alone in the universe, then in (3.23) we are to put

$$-d\tilde{s}_g^2 = \delta_{mn} dx_m dx_n = dx_n dx_n, \tag{4.1}$$

and from this it follows immediately that the world line is straight.

We seek the field of this particle, and for this field we shall get the simplest expression by using special axes in space-time with the time-axis coincident with the world line of the particle. We are then to put in (3.21)

$$\begin{aligned} \lambda'_\alpha &= 0, \lambda'_4 = i, \\ \xi_\alpha &= x_\alpha, \xi_4 = ir, \\ w &= r \end{aligned} \tag{4.2}$$

(cf. 3.13), where r is the spatial distance of the point of observation P from the fixed particle, for which $x'_\alpha = 0$. Now, m being the mass of the particle, we have

$$g_{mn} = \delta_{mn} + (mk/r^3)\xi_m \xi_n, \tag{4.3}$$

and so, with Greek suffixes for the range 1,2,3 in accordance with our convention,

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} + \frac{mk}{r^3} x_\mu x_\nu \\ g_{\mu 4} &= i\frac{mk}{r^2} x_\mu \\ g_{44} &= 1 - \frac{mk}{r} \end{aligned} \tag{4.4}$$

This gives the fundamental form

$$\begin{aligned} \Phi &= g_{mn} dx_m dx_n \\ &= g_{\mu\nu} dx_\mu dx_\nu + 2g_{\mu 4} dx_\mu dx_4 + g_{44} dx_4^2 \\ &= dx_\mu dx_\nu + (mk/r^3)(x_\mu dx_\mu)^2 + 2i(mk/r^2) x_\mu dx_\mu dx_4 + (1 - mk/r) dx_4^2. \end{aligned} \tag{4.5}$$

We now introduce spherical polar coordinates (Fig.I.3), so that

$$\begin{aligned} dx_\mu dx_\mu &= dr^2 + r^2 d\Omega, d\Omega \\ &= d\theta^2 + \sin\theta d\phi^{2^2}, \; x_\mu dx_\mu = r dr \end{aligned} \tag{4.7}$$

and the form Φ reads, since $x_4 = ict$,

$$\begin{aligned} \Phi &= (1 + \frac{mk}{r}) dr^2 + r^2 d\Omega \\ &- \frac{2mkc}{r} dr dt - (1 - \frac{mk}{r}) c^2 dt^2 \end{aligned} \tag{4.8}$$

This may be called the *Whitehead form for the field of a particle at rest.*

Eddington (Nature 113, 192 (1924)) pointed out a remarkable fact: we can transform the Whitehead form into the Schwarzschild form which occurs in the General Theory of Relativity! This is certainly surprising in view of the difference between Whitehead's simple definition of the field as in (3.21) and the Einstein procedure which involves the solution of non-linear partial differential equations.

To carry this out, let us make a transformation from (r, θ, ϕ, t) to (r, θ, ϕ, τ) of the form

$$ct = c\tau + f(r), \tag{4.9}$$

f being a function to be determined later. We get from (4.8)

$$\Phi = (1+\frac{mk}{r})dr^2+r^2d\Omega-\frac{2mk}{r}dr(cd\tau+f'(r)dr)-(1-\frac{mk}{r})(cd\tau+f'(r)dr)^2, \; (2)$$

and this has the following coefficients:

$$\begin{aligned} &\text{of } dr^2 : 1+\frac{mk}{r}-\frac{2mk}{r}f'(r) \\ &\qquad -(1-\frac{mk}{r})[f'(r)]^2, \qquad (4.11) \\ &\text{of } 2cdrd\tau : -\frac{mk}{r}-f'(r)(1-\frac{mk}{r}), \\ &\text{of } c^2d\tau^2 : -(1-\frac{mk}{r}). \end{aligned}$$

If we can make

$$f'(r) = -\frac{mk/r}{1-\frac{mk}{r}} = -\frac{mk}{r-mk}, \tag{4.12}$$

then the term in $drd\tau$ vanishes, and we get

$$\begin{aligned} \Phi = {} & (1-mk/r)^{-1}dr^2 + r^2d\Omega \\ & -(1-mk/r)c^2d\tau^2, \end{aligned} \tag{4.13}$$

which is precisely the Schwarzschild form for the solar field, i.e. the case of radial symmetry.

But can we satisfy (4.12)? We must remember that in the whole of space r has the range $0 < r < \infty$, and on this account a difficulty appears. The solution of (4.12) is in fact discontinuous, with infinite value, for $r = mk$, since (4.12) gives on integration

$$\begin{aligned} f(r) &= mk \ln \frac{mk}{mk-r} + A \text{ for } r < mk, \qquad (\quad 4.14) \\ f(r) &= mk \ln \frac{mk}{r-mk} + B \text{ for } r > mk, \end{aligned}$$

A and B being constants of integration. Thus we can carry out the transformation to the Schwarzschild form (4.13) either inside the "singularity" $r = mk$ or outside it, but not for both regions at the same time. Either of (4.14) makes τ infinite for $r = mk$.

The value $r = mk$ corresponds to the so-called “Schwarzschild singularity” of the General Theory of Relativity, for we shall later identify the universal constant k with $2G/c^2$ where G is the constant of gravitation. We note that the Whitehead form (4.8) is free from singularity, except of course at $r = 0$, where we expect it. The Schwarzschild singularity makes its appearance through the attempt to carry out a transformation to get rid of the product term in $drdt$; it is, from the Whitehead standpoint, an artificial mathematical singularity which it is .unnecessary to introduce. Nevertheless the value $r = mk$ is critical in the Whitehead Theory, as we shall see in Lecture III.

3.2 Lecture II: CELESTIAL MECHANICS

3.2.1

3.2.2 2.1 Equations of orbits

On account of the transformability of the Whitehead form into the Schwarzschild form, and on account of the fact that the equations of particle orbits are essentially the same, viz., in our notation

$$\delta \int d\overline{s}_g = 0, \tag{1.1}$$

it is to be concluded that planetary orbits in the Whitehead field of a fixed central particle are transforms of the orbits in the Schwarzschild field. That is in fact true, but it seems worth while to discuss the Whitehead orbits directly. Note that for the present we are dealing with the fiield of a massive *particle*; later we shall consider the field of a *sphere* of finite size.

The differential equations corresponding to (1.1) may be written in the form

$$\frac{d^2 x_r}{d\overline{s}_g{}^2} + \begin{Bmatrix} r & \\ m & n \end{Bmatrix} \frac{dx_m}{d\overline{s}_g} \frac{dx_n}{d\overline{s}_g} = 0, \qquad ((1.2))$$

where $\begin{Bmatrix} r & \\ m & n \end{Bmatrix}$ is a Christoffel symbol, but it is easier to use a Lagrangian form. Let us write

$$2L = (1 - \frac{mk}{r}) r'^2 + r^2(\theta'^2 + \sin^2\theta \phi'^2) \tag{1.3}$$

$$- \frac{2mkc}{r} r't' - (1 - \frac{mk}{r}) c^2 t'^2.$$

(3)

where the prime means differentiation with respect to $\overline{s}_g$ as parameter. Then (1.2) is equivalent to

$$\frac{d}{d\overline{s}_g} \frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = 0, \text{ etc.,} \tag{1.4}$$

and we have the first integral

$$2L = -1. \tag{1.5}$$

Since ϕ and t are ignorable, we have the first integrals

$$\frac{\partial L}{\partial \phi'} = r^2 \sin^2\theta \phi' = \alpha, \tag{1.6}$$

$$\frac{\partial L}{\partial t'} = -mkcr'/r - (1 - mk/r)c^2 t' = -\beta c,$$

where α and β are constants, α having the dimensions of a length, and β *zero* dimensions.

The equation

$$\frac{d}{d\overline{s}_g}\frac{\partial L}{\partial e'} - \frac{\partial L}{\partial \theta} = \frac{d}{d\overline{s}_g}(r^2\theta') - r^2\sin\theta\cos\theta{\phi'}^2 = 0 \tag{1.7}$$

tells us that if we choose axes in space so that initially $\theta = \frac{1}{2}\pi, \quad \theta' = 0$ (as we surely can), then $\theta = \frac{1}{2}\pi$ permanently, so that the orbit is plane. We put $\theta = \frac{1}{2}\pi$ and investigate the three equations

$$\begin{aligned} r^2\phi' = \alpha, \quad mkr'/r + (1 - mk/r)ct' = \beta, \\ -1 = (1 + mk/r)^2){r'}^2 + r^2{\phi'}^2 - 2mkcr't/r - (1 - mk/r)c^2{t'}^2. \end{aligned} \tag{1.8}$$

Multiplying the last of these by $(1 - mk/r)$ and using the second, we get

$$(1-m^2k^2/r^2){r'}^2+(1-mk/r)r^2{\phi'}^2-2mkr'(\beta-mkr'/r)/r-(\beta-mkr'/r)^2 = -(1-mk/r), \tag{1.9}$$

or

$${r'}^2 + (1 - mk/r)r^2{\phi'}^2 - \beta^2 = -(1 - mk/r). \tag{1.10}$$

We now use the first of (1.8) to get an equation homogeneous in r' and ϕ':

$${r'}^2 + (1 - mk/r)r^2{\phi'}^2 - (\beta^2/\alpha^2)r^4{\phi'}^2 + (1/\alpha^2)r^4{\phi'}^2(1 - mk/r) = 0. \tag{1.11}$$

Then, putting $u = 1/r, \quad r'/\phi' = dr/d\theta = -u^{-2}du/d\phi$, and dividing (1.11) by ${\phi'}^2$, we have

$$u^{-4}(du/d\phi)^2 + (1 - mku)u^{-2} - (\beta^2/\alpha^2)u^{-4} + (1/\alpha^2)(1 - mku)u^{-4} = 0, \tag{1.12}$$

or

$$(du/d\phi)^2 = f(u), \tag{1.13}$$

where

$$f(u) = \beta^2/\alpha^2 - (1 - mku)/\alpha^2 - u^2(1 - mku). \tag{1.14}$$

Since this is a cubic in u, (1.13) gives u as an elliptic function of ϕ, so the orbits are determined.

To compare with Newtonian theory, we differentiate (1.13) and obtain

$$d^2u/d\phi^2 = \frac{1}{2}f'(u) = \frac{1}{2}mk/\alpha^2 - u + (3/2)mku^2, \tag{1.15}$$

or

$$d^2u/d\phi^2 + u = \frac{1}{2}mk/\alpha^2 + (3/2)mku^2. \tag{1.16}$$

Now for slow motion and weak fields (the conditions under which we expect approximation to the Newtonian theory), from (1.3), we have approximately

$$2L = -c^2{t'}^2 = -1, \quad d\overline{s}_g = cdt,$$

and (1.6) gives

$$r^2\phi' = r^2(d\phi/dt)(dt/d\overline{s}_g) = c^{-1}r^2 d\phi/dt = \alpha,\ r^2 d\phi/dt = \alpha c,$$

so that αc is approximately the angular momentum per unit mass. Denoting this by h in the usual Newtonian notation, so that

$$h = r^2 d\phi/dt,$$

we may write (1.16) approximately as

$$d^2u/d\phi^2 + u = \frac{1}{2}mkc^2/h^2 + (3/2)mku^2, \tag{1.17}$$

in which the last term is relatively very small. This is to e compared with the Newtonian equation

$$d^2u/d\phi^2 + u = Gm/h^2, \tag{1.18}$$

where G is the gravitational constant, and we bring the Whitehead theory into limiting coincidence with Newtonian theory for slow motion and weak fields by making the following choice of the universal constant k:

$$k = 2G/c^2. \tag{1.19}$$

3.2.3 2.2. Rotation of the perihelion

Since (1.13) and (1.14) are formally the same as the equations which occur in the Einstein theory, we can investigate the rotation of the perihelion of an orbit in the same way for both theories. The following seems to be simplest.

The apsides of an orbit (stationary values of r) occur for $du/d\phi = 0$, or equivalently $f(u) = 0$, where f is as in (1.14). Write

$$f(u) = mk(u-u_1)(u-u_2)(u-u_3), \tag{4}$$
$$u_1 + u_2 + u_3 = 1/(mk) = \frac{1}{2}c^2/(mG),$$

this last being a large number. We identify u_1 and u_2 with the reciprocals of the two apsidal distances of the orbit; then the third zero of $f(u)$, namely u_3, is large.

By (1.13) the apsidal angle is accurately

$$A = \int_{u_1}^{u_2} [f(u)]^{-1/2} du \tag{2.2}$$
$$= (mk)^{-1/2} \int_{u_1}^{u_2} [(u_2-u)(u-u_1)(u_3-u)]^{-1/2} du,$$

if we take $u_1 < u_2$. Hence approximately

$$A = (mku_3)^{-1/2} \int_{u_1}^{u_2} [(u_2-u)(u-u_1)]^{-1/2} (1 + \frac{u}{2u_3}) du. \tag{2.3}$$

But

$$\int_{u_1}^{u_2} [(u_2 - u)(u - u_1)]^{-1/2} du = \pi, \tag{2.4}$$

$$\int_{u_1}^{u_2} [(u_2 - u)(u - u_1)]^{-1/2} u du = \frac{1}{2}\pi(u_1 + u_2)$$

$$(mku_3)^{-1/2} = [1 - mk(u_1 + u_2)]^{-1/2} = 1 + \frac{1}{2}mk(u_1 + u_2), \text{ approx.},$$

$$1/u_3 = 1/[(mk)^{-1} - u_1 - u_2] = mk, \text{ approx.},$$

and so

$$\begin{aligned} A &= [1 + \frac{1}{2}mk(u_1 + u_2)][\pi + \frac{1}{4}\pi mk(u_1 + u_2)] \\ &= \pi[1 + \frac{3}{4}mk(u_1 + u_2)]. \end{aligned} \tag{2.5}$$

Thus the advance of perihelion per orbital revolution is

$$2A - 2\pi = \frac{3}{2}\pi mk(u_1 + u_2) = 3\pi mGc^{-2}(u_1 + u_2), \tag{2.6}$$

and if we denote the semi-axis major of the orbit by a and the eccentricity by e, so that

$$\begin{aligned} 1/u_1 &= a(1 + e), \quad 1/u_2 = a(1 - e), \\ u_1 + u_2 &= 2/a(1 - e^2), \end{aligned} \tag{2.7}$$

then the advance is

$$2A - 2\pi = \frac{6\pi mG}{ac^2(1 - e^2)}. \tag{2.8}$$

If, finally, we put $mG = 4\pi^2 a^3/t^2$, where t is the periodic time, we obtain the now classical formula

$$2A - 2\pi = \frac{24\pi^3 a^2}{c^2 t^2 (1 - e^2)}. \tag{2.9}$$

This formula for the rotation of perihelion is common to Einstein's General Theory of Relativity and to Whitehead's theory of gravitation. The only difference lies in different interpretations of r and t, and of course these differences are significant in the case of planetary orbits.

3.2.4 2.3 Circular orbits

Let us discuss circular orbits in the field of a fixed particle. We have to use not only the first-order equation (1.13), which reads

$$(du/d\phi)^2 = \beta^2/\alpha^2 - (1/\alpha^2)(1 - mku) - u^2(1 - mku), \tag{3.1}$$

but also the second-order equation (1.15) obtained by differentiating it:

$$d^2u/d\phi^2 = \frac{1}{2}mk/\alpha^2 - u + (3/2)mku^2. \tag{3.2}$$

Proceeding without approximation, we note the presence of two constants:

an angular momentum constant

$$\alpha = r^2\phi', \tag{3.3}$$

and an energy constant

$$\beta = mkr'/r + (1 - mk/r)ct', \tag{3.4}$$

where the prime indicates $d/d\overline{s}_g$.

Circular orbits for given α are found by putting $d^2u/d\phi^2 = 0$ in (3.2); thus the radius $r = 1/u$ is to satisfy the quadratic equation

$$r^2 - 2\alpha^2 r/(mk) + 3\alpha^2 = 0, \tag{3.5}$$

or, since $k = 2G/c$,

$$r^2 - \alpha^2 c^2 r/(mG) + 3\alpha^2 = 0, \tag{3.6}$$

of which the roots are

$$r = \frac{1}{2}[\frac{\alpha^2 c^2}{mG} \pm \sqrt{\frac{\alpha^4 c^4}{m^2 G^2} - 12\alpha^2}]. \tag{3.7}$$

Thus there exists circular orbits only for values of α satisfying

$$\alpha^2 \geq 12m^2G^2/c^4 = 3m^2k^2. \tag{3.8}$$

So there is a lower bound for possible angular momentum in a circular orbit.

By (3.6) we also have

$$\alpha^2 = \frac{r^2}{\frac{rc^2}{mG} - 3}, \tag{3.9}$$

which tells us that

$$r > 3mG/c^2 = (3/2)mk. \tag{3.10}$$

For any r satisfying this inequality there exists a circular orbit, with α^2 given by (3.9). For any α^2 satisfying (3.8) there exist two circular orbits except that for any $\alpha^2 = 12m^2G^2/c^4$ there is only one, with radius $r = 6mG/c^2$.

The general form of the graph connecting r and α for circular orbits is shown in Figure II.1.

If we use (3.3) and (3.4), with $r' = 0$ in the latter, the angular velocity in a circular orbit is

$$\omega = \frac{d\phi}{dt} = \frac{\alpha}{\beta}\frac{c}{r^2}(1 - \frac{mk}{r}) = \frac{\alpha}{\beta}\frac{c}{r^2}(1 - \frac{2Gm}{c^2 r}). \tag{3.11}$$

If we put $du/d\phi = 0$ in (3.1), we get

$$\frac{\beta^2}{\alpha^2} = (\frac{1}{\alpha^2} + u^2)(1 - mku) = (\frac{1}{\alpha^2} + \frac{1}{r^2})(1 - \frac{mk}{r}), \tag{3.12}$$

and so by (3.9)

$$\frac{\beta^2}{\alpha^2} = (\frac{c^2}{mGr} - \frac{3}{r^2} + \frac{1}{r^2})(1 - \frac{2mG}{c^2 r}) = \frac{c^2}{mGr}(1 - \frac{2mG}{c^2 r})^2. \tag{3.13}$$

When we substitute this in (3.11) we find that the angular velocity in a circular orbit satisfies precisely the Newtonian equation

$$\omega^2 = mG/r^3. \tag{3.14}$$

In Whitehead's relativity, as in Einstein's General Theory, the concept of force is abandoned in favor of a variational principle as in (1.1). We can however introduce it in Whitehead's theory in a rather special way in connection with circular orbits, by taking an analogy with Newtonian mechanics. For in Newtonian mechanics the force per unit mass on a particle describing a circular orbit of radius r with angular velocity admits two "definitions":

> (a) by means of angular velocity: $F_\omega = \omega^2 r$; (b) by means of angular momentum: $F_h = h^2/r^3$.

Since $h = \omega r^2$, we have $F_\omega = F_h$.

If we carry the analogy into Whitehead's relativity, and use α_c instead of h, we get the following definitions:

> (a) by means of angular velocity: $F_\omega = \omega^2 r = mG/r^2$ by (3.14); (b) by means of angular momentum: $F_\alpha = \frac{\alpha^2 c^2}{r^3} = \frac{mG}{r^2}(1 - \frac{3mG}{c^2 r})^{-1}$.

Thus F_ω agrees with the Newtonian value, whereas F_α tends to infinity as r tends to $3mG/c^2$, which is the radius of the smallest possible circular orbit.

3.2.5 2.4. The gravitational field of a sphere

Suppose we have a finite body at rest. Then, referring to (3.22) and (4.4) of Lecture I, the field at the point x_μ is given by

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu nu} + k \sum m'(x_\mu - x'_\mu)(x_\nu - x'_\nu)/r^3, \\ g_{\mu 4} &= ik \sum m'(x_\mu - x'_\mu)/r^2, \\ g_{44} &= 1 - k \sum m'/r, \end{aligned} \tag{4.1}$$

where $k = 2G/c^2$ and the summations are over all the particles forming the body, m' being a typical mass, x'_μ a typical position, and $r^2 = (x_\mu - x'_\mu)(x_\mu - x'_\mu)$. We recall that Greek suffixes have the range 1, 2, 3.

For a continuous distribution of density $\rho(x')$, we replace the summations by integrations:

$$g_{\mu\nu} = \delta_{\mu\nu} + k\int \rho(x')(x_\mu - x'_\mu)(x_\nu - x'_\nu)r^{-3}d\tau' \tag{4.2}$$

$$g_{\mu 4} = ik\int \rho(x')(x_\mu - x'_\mu)r^{-2}d\tau'$$

$$g_{44} = 1 - k\int \rho(x')r^{-1}d\tau',$$

$d\tau'$ being the element of volume.

Suppose now that the sphere is of uniform density, with radius a and center at the origin. We have to evaluate the integrals in (4.2), with ρ removed from the integrands, since it is constant.

From consideration of tensor form it is clear that, if we write

$$I_{\mu\nu} = \int (x_\mu - x'_\mu)(x_\nu - x'_\nu)r^{-3}d\tau', \tag{4.3}$$

$$I_\mu = \int (x_\mu - x'_\mu)r^{-2}d\tau',$$

then these integrals may be expressed in the form

$$I_{\mu\nu} = f(r_O)x_\mu x_\nu + g(r_O)\delta_{\mu\nu}, \quad I_\mu = h(r_O)x_\mu, \tag{4.4}$$

where f, g, h are three functions (which we have to evaluate) and r_O is the distance from the center of the sphere to the point x_μ, so that

$$r_O^2 = x_\mu x_\mu. \tag{4.5}$$

To find these three functions f, g, h, we take the point x_μ on the x_3-axis, so that

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = r_O. \tag{4.6}$$

Then, in terms of spherical polar coordinates R, θ, ϕ, the formulae (4.3) give for $I_{\mu\nu}$

$$I_{11} = I_{22} = \int_{R=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{R^4 \sin^3\theta \cos^2\theta dR d\theta d\phi}{(R^2 + r_O^2 - 2Rr_O\cos\theta)^{3/2}}$$

the others vanishing, and for I_μ

$$I_3 = \int_{R=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{(r_O - R\cos\theta)R^2\sin\theta dR d\theta d\phi}{R^2 + r_O^2 - 2Rr_O\cos\theta}, \tag{4.8}$$

the other two vanishing.

Direct integration gives, for exterior points ($r_O > a$),

$$I_{11} = I_{22} = (4\pi/15)a^5 r_O^{-3}. \tag{4.9}$$

$$I_{33} = (4\pi/3)a^3 r_O^{-1}(1 - \frac{2}{5}a^2 r_O^{-2})$$
$$= (4\pi/3)a^3 r_O^{-1} - (8\pi/15)a^5 r_O^{-3};$$

then by (4.4)

$$I_{11} = g(r_O) = (4\pi/15)a^5 r_O^{-3}, \tag{4.10}$$
$$I_{33} = f(r_O)r_O^2 g(r_O) = (4\pi/3)a^3 r_O^{-1} - (8\pi/15)a^5 r_O^{-3},$$

and so

$$f(r_O) = (4\pi/3)a^3 r_O^{-3}(1 - \frac{3}{5}a^2 r_O^{-2}), \tag{4.11}$$
$$g(r_O) = (4\pi/15)a^5 r_O^{-3}.$$

When these functions are substituted in (4.4), we have the general expression for $I_{\mu\nu}$ for any exterior point x_μ.

Also, from (4.8),

$$I_3 = -\frac{1}{4}\pi r_O^2\{(1 - a^2/r_O^2)^2 \log\frac{1 + a/r_O}{1 - a/r_O} - 2\frac{a}{r_O}(1 + \frac{a^2}{r_O^2})\}. \tag{4.12}$$

(If the ratio a/r_O is small, this complicated expression reduces to the approximate value $\frac{4\pi}{3}a^3 r_O^{-1}$.) By (4.4) we have

$$I_3 = h(r_O)r_O, \tag{4.13}$$

and so comparison of (4.12) and (4.13) gives

$$h(r_O) = \frac{1}{4}\pi r_O\{2\frac{a}{r_O}(1 + \frac{a^2}{r_O^2}) - (1 - \frac{a^2}{r_O^2})^2 \log\frac{1 + a/r_O}{1 - a/r_O}\} \tag{4.14}$$
$$(= \frac{4\pi}{3}a^3 r_O^{-2} \text{ approximately, for small } a/r_O).$$

The value of I_μ for any exterior point x_μ is given by substituting this function in (4.4).

The integral for g_{44} in (4.2) is of course elementary when ρ is a constant, and thus ,the evaluation of (4.2) is complete, for we have

$$g_{\mu\nu} = \delta_{\mu\nu} + k\rho I_{\mu\nu}, \quad g_{\mu 4} = ik\rho I_\mu, \quad g_{44} = 1 - km/r_O, \tag{4.15}$$

where m is the total mass of the sphere:

$$m = \frac{4\pi}{3}\rho a^3. \tag{4.16}$$

The quadratic form for the field is

$$\begin{aligned}\Phi &= g_{mn}dx_m dx_n = g_{\mu\nu}dx_\mu dx_\nu + 2ig_{\mu 4}dx_\mu cdt - g_{44}c^2dt^2 \\ &= dx_\mu dx_\mu + k\rho I_{\mu\nu}dx_\mu dx_\nu - 2k\rho I_\mu dx_\mu cdt - (1-mk/r_O)c^2dt^2 \\ &= [1+k\rho g(r_O)]dx_\mu dx_\mu + k\rho f(r_O)(x_\mu dx_\mu)^2 - 2k\rho h(r_O)x_\mu dx_\mu cdt - (1-mk/r_O)c^2dt^2.\end{aligned} \tag{4.17}$$

Let us now for simplicity write r for r_O, so that henceforth r is the distance from the center of the sphere:

$$r^2 = x_\mu x_\mu. \tag{4.18}$$

Then

$$\begin{aligned}k\rho g(r) &= (4\pi/15)k\rho a^5 r^{-3} = (1/5)kma^2r^{-3}, \\ k\rho f(r) &= (4\pi/3)k\rho a^3 r^{-3}(1-\frac{3}{5}a^2/r^2) = kmr^{-3}(1-\frac{1}{5}\frac{a^2}{r^2}), \\ k\rho h(r) &= \frac{3}{16}km\frac{r}{a^3}\{2\frac{a}{r}(1+\frac{a^2}{r^2} - (1-\frac{a^2}{r^2})\log\frac{1+a/r}{1-a/r},\end{aligned} \tag{4.19}$$

or approximately for small a/r

$$k\rho h(r) = kmr^{-2}. \tag{4.20}$$

Then, introducing spherical polar coordinates r, θ, ϕ with $d\Omega = d\theta^2 + \sin\theta^2 d\phi^2$, we have from (4.17)

$$\begin{aligned}\Phi &= [1+k\rho g(r)][dr^2 + r^2 d\Omega] + k\rho f(r)r^2dr^2 - 2k\rho h(r)rdrcdt- \\ &\quad (1-mk/r)c^2dt \\ &= Adr + Br^2d\Omega - 2Cdr\cdot cdt - (1-km/r)c^2dt^2\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}A &= 1 + k\rho[g(r) + r^2 f(r)] \\ B &= 1 + k\rho g(r), \quad C = k\rho rh(r),\end{aligned} \tag{4.22}$$

or explicitly,

$$\begin{aligned}A &= 1 + \frac{km}{r}(1-\frac{2}{5}\frac{a^2}{r^2}), \\ B &= 1 + \frac{1}{5}km\frac{a^2}{r^3} \\ C &= \frac{3}{16}km\frac{r^2}{a^3}\{2\frac{a}{r}(1+\frac{a^2}{r^2}) - (1-\frac{a^2}{r^2})^2\log\frac{1+a/r}{1-a/r}\}\end{aligned} \tag{4.23}$$

(for small a/r, $C = km/r$ approximately).

We have then in (4.21) the quadratic form expressing *the gravitational field of a sphere of uniform density, at rest.* It should be compared with (4.8) of

Lecture I, which gives the gravitational field of a *particle*. We see at once that the field does not depend solely on the total mass (m) of the sphere, as it does in Newtonian gravitation, for the radius a of the sphere appears in the coefficients A, B, C above. The radius does not, however, appear in the coefficients of dt^2 in (4.21), and since this term is the most important dynamically, the effects due to appearance of the radius in the other coefficients will be very small. We note that A and B differ from the corresponding coefficients for the field of a particle by quantities of the order kma^2r^{-3} and this difference is very small unless the point of observation is close to the surface of the sphere. This is also true of C, which has a curiously complicated form; expansion in powers of a/r gives

$$C = \frac{km}{r}(1 - \frac{1}{5}a^2r^{-2} + O(a^4r^{-4})). \tag{4.24}$$

Planetary orbits in the field of a finite spherical sun of constant density can of course be worked out. The method is that of Section 1, and instead of (1.13) we find, for $u = 1/r$, the differential equation

$$(du/d\phi)^2 = F(u) \tag{4.25}$$

where

$$F(u) = \frac{[\beta^2 - (1 - kmu)]B^2/\alpha^2 - (1 - kmu)Bu^2}{A(1 - kmu) + C^2} \tag{4.26}$$

α and β being constants of integration:

$$Br^2\phi' = \alpha \tag{4.27}$$

$$Cr' + (1 - km/r)ct' = \beta \tag{5}$$

where the prime indicates differentiation with respect to $\overline{s}_g$.

It is possible also to find the gravitational field of a *rotating* sphere and discuss the orbits of planets in this field. In this case however retardation plays a part, and it is necessary to have recourse to approximations which make the work somewhat clumsy. No further account of this problem will be given here.

3.2.6 2.5. The two-body problem

In Whitehead's theory it is possible to set up without difficulty the equations governing the motion of two particles under the influence of their mutual gravitational interaction. On account of the retardation involved, they are differential-difference equations, such as one meets in the electromagnetic two-body problem.

Let L and L' be the two world lines (Figure II.2). We take any event P on L and any event P' on L' and draw from them null cones into the past. Then as in Lecture I,

$$(g_{mn})_P = \delta_{mn} + km'w_P^{-3}(\xi_m\xi_n)_P \tag{5.1}$$

$$w_P = -(\xi_n)_P(\lambda_n)_{P'_O}$$

$$(\xi_n)_P = (x_n)_P - (x_n)_{P'_O}, \quad (\lambda_n)_{P'_O} = (dx'_n/ds\prime)_{P'_O}$$

and the field at P' due to L is

$$\begin{aligned}(g_{mn})_{P'} &= \delta_{mn} + kmw_{P'}^{-3}(\xi_m\xi_n)_{P'} \\ w_{P'} &= -(\xi_n)_{P'}(\lambda_n)_{P_O} \\ (\xi_n)_{P'} &= (x_n)_{P'} - (x_n)_{P_O}, \quad (\lambda_n)_{P_O} = (dx_n/ds)_{P_O}.\end{aligned} \tag{5.2}$$

The variational principles which define the motion read

$$\delta \int d\overline{s}_g = 0, \quad \delta \int d\overline{s}'_g = 0, \tag{5.3}$$

and these give the differential equations of motion of the second order:

$$\begin{aligned}\frac{d^2x_r}{d\overline{s}_g^2} + \begin{Bmatrix} r & \\ m & n \end{Bmatrix}_P \frac{dx_m}{d\overline{s}_g}\frac{dx_n}{d\overline{s}_g} &= 0, \\ \frac{d^2x'_r}{d\overline{s}'^2_g} + \begin{Bmatrix} r & \\ m & n \end{Bmatrix}_{P'} \frac{dx'_m}{d\overline{s}'_g}\frac{dx'_n}{d\overline{s}'_g} &= 0.\end{aligned} \tag{5.4}$$

However the calculation of the Christoffel symbol is tedious, and it is easier to work with Lagrangian forms, writing

$$\begin{aligned}2\Lambda_P(x, dx) &= (g_{mn}dx_m dx_n)_P \\ &= (dx_n dx_n)_P + km'w_P^{-3}(\xi_n dx_n)^2_P.\end{aligned} \tag{5.5}$$

Let us simplify the notation by dropping the subscript P. Thus we write

$$2\Lambda(x, dx) = dx_n dx_n + km'w^{-3}[(x_n - x'_n)dx_n]^2, \tag{5.6}$$

wherein w and x'_n are to be regarded as functions of x'_n, obtained by drawing the null cone into the past from P and using its intersection with the world line L'. Let a dot indicate differentiation with respect to $\overline{s}_g$. Then we may write (5.6) in the form

$$2\Lambda(x, \dot{x}) = \dot{x_n}\dot{x_n} + km'w^{-3}[(x_n - x'_n) \cdot x_n]^2 \tag{5.7}$$

and with a similar expression for Λ' we can express the equations of motion (5.4) in the Lagrangian form

$$\frac{d}{d\overline{s}_g}\frac{\partial\Lambda}{\partial\dot{x_n}} - \frac{\partial\Lambda}{\partial x_n} = 0, \frac{d}{d\overline{s}'_g}\frac{\partial\Lambda'}{\partial\dot{x_n}'} - \frac{\partial\Lambda'}{\partial x'_n} = 0. \tag{5.8}$$

These have the first integrals

$$2\Lambda = -1, \quad 2\Lambda' = -1. \tag{5.9}$$

Rather more explicitly, the first of (5.8) reads

$$\frac{d}{d\bar{s}_g}\{\dot{x_n} + km'w^{-3}(x_n - x'_n)(x_m - x'_m)\dot{x_m}\} \tag{6}$$

$$- km'w^{-3}(x_m - x'_m)\dot{x_m}\dot{x_p}(\delta_{pn} - \partial x'_p/\partial x_n) \tag{7}$$

$$-km'[(x_m - x'_m)\dot{x_m}]^2 \frac{\partial}{\partial x_n}(w^{-3}) = 0. \tag{8}$$

[]This equation and its companion corresponding to the second of (5.8), are not easy to handle, and anything in the nature of a "general solution" is out of the question. However, there are two problems of special simplicity to which attention might be given.

The first of these problems is the quasi-Kepler problem, in which one of the two particles is much more massive than the other. The orbit of the lighter particles approximates the Kepler orbit discussed in Sections 1 and 2. It should be possible to push the approximation further without too great labor, particularly if the relative velocity is assumed small.

The second problem is the symmetric two-body problem, in which not only are the masses of the two particles assumed to be equal but a further symmetry is imposed by suitable initial conditions- viz., as judged by some Galileian observer, the straight line joining simultaneous positions of the two particles lies in a fixed plane and has its middle point fixed. It seems likely that under these conditions the distance between the two particles steadily increases; it would be interesting to have a solution.

3.3 Lecture III: ELECTROMAGNETISM

3.3.1 3. 1. The field quantities

As always in the Whitehead theory, the background is the flat space-time of Minkowski; in it we use the imaginary time-coordinate $x_4 = ict$. In this flat space-time there is a gravitational field specified by g_{mn}, but we are not now concerned with the way in which this field is produced– it is a given field as far as electromagnetism is concerned.

We shall discuss only electromagnestism in vacuo, with special references to the propagation of light.

The electromagnetic field is described by two skew-symmetric tensors, or six-vectors; we shall denote them by

$$F_{mn} = -F_{nm}, \quad F^{mn} = -F^{nm}. \tag{1.1}$$

Here the subscript-superscript notation has no particular meaning– it is simply a notation to distinguish one set of quantities from another set, the two sets being for the present quite unrelated.

We now make the following "physical identification":

$$\begin{aligned} &F_{23} = B_1, F_{31} = B_2, F_{12} = B_3(\vec{B} = \text{magnetic induction}) \qquad (1.2)\\ &F_{14} = -iE_1, F_{24} = -iE_2, F_{34} = -iE_3(\vec{E} = \text{electric field strength})\\ &F^{23} = H_1, F^{31} = H_2, F^{12} = H_3(\vec{H} = \text{magnetic field strength})\\ &F^{14} = -iD_1, F^{24} = -iD_2, F^{34} = -iD_3(\vec{D} = \text{dielectric displacement}) \end{aligned}$$

We may also include a 4-vector J^+, with

$$\begin{aligned} &J^\alpha = (4\pi/c)j^\alpha(\vec{j} = \text{current density}) \qquad (1.3)\\ &J^4 = 4\pi i\rho(\rho = \text{charge density}) \end{aligned}$$

As previously, Latin suffixes have the range 1, 2, 3, 4 and Greek suffixes the range 1, 2, 3.

3.3.2 3.2. Maxwell's equations

We now accept the following partial differential equations (Maxwell's Equations):

$$F_{[mn,r]} = 0, \qquad F^{rs}_{,s} = J^r. \tag{2.1}$$

Here the comma denotes partial differentiation, and

$$F_{[mn,r]} = F_{mn,r} + F_{nr,m} + F_{rm,n}.$$

With the identifications (1.2) and (1.3), it is easy to see that these are in fact the usual Maxwell's equations:

$$\begin{aligned}
&F_{[23,1]} = 0 \text{ is equivalent to } div\vec{B} = 0; \qquad (2.2)\\
&F_{[23,4]} = 0, \text{ etc. are equivalent to } c^{-1}\frac{\partial \vec{B}}{\partial t} + rot\vec{E} = 0\\
&F^{\alpha s}_{,s} = J^{\alpha} \text{ equivalent to } c^{-1}\frac{\partial \vec{D}}{\partial t} - rot\vec{H} = \frac{-4\pi\vec{j}}{c},\\
&F^{4s}_{,s} = J^{4} \text{ is equivalent to } div\vec{D} = 4\pi\rho.
\end{aligned}$$

3.3.3 3. Structural equations

The number of equations in (2.1) is less than the number of field quantities (1.1) (J^{τ} being supposed given). To complete the system, we introduce what may be called *strucrural* equations as follows, involving the gravitational field:

$$F_{mn} = g_{mr}g_{ns}F^{rs}. \qquad (3.1)$$

These are linear equations expressing $\vec{B}$ and $\vec{E}$ in terms of $\vec{H}$ and $\vec{D}$. If there is no gravitational field, then $g_{mn} = \delta_{mn}$, and (3.1) become

$$F_{mn} = F^{mn}, \qquad (3.2)$$

or

$$\vec{B} = \vec{H}, \quad \vec{D} = \vec{E}, \qquad (3.3)$$

as of course is proper.

The gravitational field plays the part of dielectric constant and magnetic permeability, but more generally, since (3.1) does not separately express $\vec{D}$ in terms of $\vec{E}$ and $\vec{B}$ in terms of $\vec{H}$.

A word with regard to tensor character. In Whitehead's theory we do not seek invariance of form with respect to general transformations in space-time, but only with respect to Lorentz transformations. For the Minkowskian coordinates x_n, the Lorentz transformation leaves $dx_n dx_n$ invariant, and so is (formally) an orthogonal transformation. This implies that the transformation laws for covariant and contravariant tensors are the same, and we do not need the notation of subscripts and superscripts to distinguish them. Thus in general symbols such as A_{mn} and A^{mn} will denote tensors unrelated to one another except for whatever connection we may deliberately set up, just as we set up in (3.1) a connection between F_{mn} and F^{mn}.

We now define g^{mn} by

$$g^{mr}g_{ms} = \delta^{r}_{s} \qquad (3.4)$$

Then (3.1) may be written equivalently

$$F^{mn} = g^{mr}g^{ns}F_{rs}. \qquad (3.5)$$

We may collect our formulae as follows:

$$F_{[mn,r]} = 0, \quad F^{rs}_{,s} = J^r, \qquad (3.6)$$
$$F^{mn} = g^{mr} g^{ns} F_{rs}.$$

We find it convenient to introduce another tensor defined by

$$F^{*^{rs}} = -\frac{1}{2} i \epsilon^{rsmn} F_{mn}, \qquad (3.7)$$

where ϵ^{rsmn} (we may also write it ϵ_{rsmn}) is the usual permutation symbol, with value 0 unless the suffixes are distinct, 1 if the suffixes form the set 1234 or an even permutation thereof, -1 if the suffixes form an odd permutation of 1234. Explicitly, (3.7) read

$$F^{*^{23}} = -F_{14}, \quad F^{*^{31}} = -iF_{24}, \quad F^{*^{12}} = -iF_{34}, \qquad (3.8)$$
$$F^{*^{14}} = -iF_{23} \quad F^{*^{24}} = -iF_{31}, \quad F^{*^{34}} = iF_{12},$$

and they are equivalent to

$$F_{rs} = \frac{1}{2} i \epsilon_{rsmn} F^{*^{mn}}. \qquad (3.9)$$

We have then identically

$$F_{[23,1]} = F_{23,1} + F_{31,2} + F_{12,3} = -iF^{*^{4r}}_{,r} \qquad (3.10)$$

with similar results for other suffixes, and hence the equations $F_{[mn,r]} = 0$ of (3.6) are equivalent to $F^{*^{rs}}_{,s} = 0$. Thus we may rewrite our basic equations (3.6) in the following form, which shows only field quantities with superscripts:

$$F^{*^{rs}}_{,s} = 0, F^{rs}_{,s} = J^r, \qquad (3.11)$$
$$F^{mn} = \frac{1}{2} i g^{mr} g^{ns} \epsilon_{rspq} F^{*^{pq}}.$$

3.3.4 3.4. Geometrical optics

We consider now the propagation of electromagnetic waves (light waves), putting $J^r = 0$ in (3.11). However we shall pass at once to geometrical optics, using the plans described in the lectures on Hamilton's method. In other words, we shall deal with the characteristics.

Plan A: We assume a solution of (3.11) of the form

$$F^{mn} = G^{mn} \exp iS, \quad F^{*^{mn}} = H^{mn} \exp iS, \qquad (4.1)$$

where G^{mn} and H^{mn} are skew-symmetric and "slowly varying", whereas S is "rapidly varying". We get then, approximately,

$$F^{mn}_{,n} = iS_{,n} G^{mn} \exp iS, \quad F^{*^{mn}}_{,n} = iS_{,n} \quad H^{mn} \exp iS, \qquad (4.2)$$

and so from (3.11) we derive the system

$$G^{mn}S_{,n} = 0, \quad H^{mn}S_{,n} = 0, \tag{4.3}$$
$$G^{mn} = \frac{1}{2} i g^{mr} g^{ns} \epsilon_{rspq} H^{pq}.$$

Now $S = const.$ is the history of a phase-wave, and the partial differentiation equation satisfied by S is to be found by eliminating G^{mn} and H^{mn} from (4.3), an algebraic problem. Before attempting it, let us take up a second plan.

Plan B: Now we study the propagation of a discontinuity in the field, such as the propagation of light into darkness. It is true that the "shock conditions" across a discontinuity are not prescribed by the partial differential equations (3.11), but they are suggested by them, as a limit of the continuous case.

From (3.11) (in which we put $J^r = 0$) we have $F^{rs}_{,s} = 0$, and so, integrating through any region of space-time,

$$\int F^{mn}_{,n} dV_4 = 0. \tag{4.4}$$

Hence, by Green's theorem,

$$\int F^{mn} d \sum_n = 0, \tag{4.5}$$

where $d\sum_n$ is a directed element of the 3-space bounding V_4. Let us now flatten V_4 down on a 3-space with equation $S = const.$, and at the same time allow a discontinuity in F^{mn} to develop in the limit; we get then from (4.5) as shock condition across $S = const.$

$$\delta F^{mn} S_{,n} = 0, \tag{4.6}$$

where δF^{mn} represents the jump in F^{mn} on crossing $S = const.$ We have used the fact that $d\sum_n$ become in the limit direction ratios of the normal to $S = const.$, and so proportional to $S_{,n}$.

Thus, from the whole set (3.11), we get the set of shock conditions

$$\delta F^{mn} S_{,n} = 0, \quad \delta {F^*}^{mn} S_{,n} = 0, \tag{4.7}$$
$$\delta F^{mn} = \frac{1}{2} i g^{mr} g^{ns} \epsilon_{rspq} \delta {F^*}^{pq}.$$

To get the partial differential equation satisfied by S, we have to eliminate the quantities δF^{mn} and $\delta {F^*}^{mn}$, an algebraic problem precisely the same as that involved in (4.3).

To emphasize the purely algebraic nature of our problem, let us rewrite it in new notation: It is required to eliminate A^{mn} and B^{mn} from the equations

$$A^{mn} T_n = 0, \quad B^{mn} T_n = 0, \tag{4.8}$$
$$A^{mn} = \frac{1}{2} i g^{mr} g^{ns} \epsilon_{rspq} B^{pq},$$
$$(A^{mn} = -A^{mn}, \quad B^{mn} = -B^{mn}).$$

It may be observed that if these equations are satisfied with non-zero T_n, then

$$\det A^{mn} = 0, \quad \det B^{mn} = 0. \tag{4.9}$$

These are skew-symmetric determinants of even order, and so they are perfect squares; in fact

$$\begin{aligned}\det A^{mn} &= \begin{vmatrix} 0 & A^{12} & A^{13} & A^{14} \\ A^{21} & 0 & A^{23} & A^{24} \\ A^{31} & A^{32} & 0 & A^{34} \\ A^{41} & A^{42} & A^{43} & 0 \end{vmatrix} \\ &= (A^{23}A^{14} + A^{31}A^{24} + A^{12}A^{34})^2,\end{aligned} \tag{4.10}$$

so that (4.9) imply

$$\begin{aligned} A^{23}A^{14} + A^{31}A^{24} + A^{12}A^{34} &= 0, \\ B^{23}B^{14} + B^{31}B^{24} + B^{12}B^{34} &= 0. \end{aligned} \tag{4.11}$$

We carry out the elimination in (4.8) in two steps. First take the case of *no gravitational field*. Then $g^{mn} = \delta_{mn}$ and our equations read

$$\begin{aligned} A^{mn}T_n = 0, \quad B^{mn}T_n &= 0, \\ A^{mn} &= \frac{1}{2}i\epsilon_{mnpq}B^{pq}. \end{aligned} \tag{4.12}$$

These equations are invariant under a Lorentz transformation, if T_n transforms as a vector and A^{mn} and B^{mn} as tensors. If T_n exists (not zero in all components), it must be space-like, time-like, or null. If it is space-like, we can choose our frame of reference so that

$$T_1 \neq 0, \quad T_2 = T_3 = T_4 = 0, \tag{4.13}$$

and hence by the first line of (4.12)

$$A^{21} = A^{31} = A^{41} = 0, B^{21} = B^{31} = B^{41} = 0. \tag{4.14}$$

By the last of (4.12) these imply the vanishing of all the components of A^{mn} and B^{mn}, and so we get no wave. Similarly the case of time-like T_n must be ruled out, and we are left with the sole possibility that T_n is a null vector, so that

$$T_nT_n = 0. \tag{4.15}$$

This, then, is the result of eliminating the A's and B's from (4.12).

We now return to the general case (4.8) and carry out the elimination by a trick, using the preceding result. Consider the application of a non-singular linear transformation, not orthogonal. Let us agree that T_n is to transform as a covariant vector, g_{mn} like a covariant tensor, g^{mn} like a contravariant tensor, and B^{mn} also like a contravariant tensor. If g denotes $\det g_{mn}$, then it is known that $g^{\frac{1}{2}}\epsilon_{rspq}$ transforms liike a covariant tensor. If we finally decide to make $g^{\frac{1}{2}}A^{mn}$ transform like a covariant tensor, we see that (4.8) retain their form under the linear transformation, if they are written equivalently as

$$g^{\frac{1}{2}}A^{mn}T_n = 0, \quad B^{mn}T_n = 0, \tag{4.16}$$
$$g^{\frac{1}{2}}A^{mn} = \frac{1}{2}ig^{mr}g^{ns}g^{\frac{1}{2}}\epsilon_{rspq}B^{pq}.$$

Now we know that there exists a linear transformation $L(x \to x')$ which makes $g'^{mn} = \delta_{mn}, g' = 1$. As a result of L, (4.16) takes on the same form as (4.12), but marked with primes. But we know that from these primed equations we obtain, as in (4.15),

$$T'_nT'_n = 0, \tag{4.17}$$

or, equivalently,

$$g'^{mn}T'_mT'_n = 0. \tag{4.18}$$

But this is an invariant equation, and so if we now apply the transformation $L^{-1}(x' \to x)$ we get

$$g^{mn}T_mT_n = 0. \tag{4.19}$$

This, then, is the result of eliminating the A's and B's from (4.18).

Restoring the original notation, we see that the phase-wave of (4.1), of the shock wave of (4.7), is propagated according to the partial differential equation

$$g^{mn}S_{,m}S_{,n} = 0. \tag{4.20}$$

3.3.5 3.5. Light rays

Equation (4.20) may be regarded as the tangential equation of a surface, $S_{,n}$ being direction ratios of its normal. All surfaces satisfying (4.20) at an event envelope an elementary cone having its vertex at that event (Figure III.1). To find the equation of the cone, we denote by ξ_n an elementary generator. Then we have

$$\xi_nS_{,n} = 0, \tag{5.1}$$

and from the envelope condition,

$$\xi_n\delta S_{,n} = 0, \quad \text{provided } g^{mn}S_{,m}\delta S_{,n} = 0. \tag{5.2}$$

Hence

$$\xi_n = \theta g^{mn}S_{,m}(\theta \text{ undetermined infinitesimal}) \tag{5.3}$$

and so

$$g_{nr}\xi_n\xi_r = \theta^2g^{mp}S_{,m}S_{,p} = 0, \tag{5.4}$$

by (4.20). Thus

$$g_{mn}dx_mdx_n = 0 \tag{5.5}$$

is the equation of the elementary cone.

At each event on a wave-surface $S = const.$ there is an elementary cone which touches this wave-surface, and the directions of tangency define space-time curves (bicharacteristics). These curves we call *rays*. It follows then from (5.3) that for some parameter u a ray satisfies

$$dx_r/du = g^{rm}S_{,n} \text{ or } g_{rn}dx_n/du = S_{,r}. \tag{5.6}$$

Hence we can derive the differential equations of a light ray. We have

$$\begin{aligned}\frac{d^2x_r}{du^2} &= g^{rn}_{,p}S_{,n}\frac{dx_p}{du} + g^{rn}S_{,np}\frac{dx_p}{du} \\ &= g^{rn}_{,p}S_{,n}g^{pq}S_{,q} + g^{rn}S_{,np}g^{pq}S_{,q}.\end{aligned} \tag{5.7}$$

But by (4.20)

$$\frac{\partial(g^pqS_{,p}S_{,q})}{\partial x_n} = 2g^{pq}S_{,pn}S_{,q} + g^{pq}_{,n}S_{,p}S_{,q} = 0, \tag{5.8}$$

and so, omitting details of calculation,

$$\begin{aligned}\frac{d^2x_r}{d\omega^2} &= g^{rn}_{,p}g^{pq}S_{,n}S_{,q} - \frac{1}{2}g^{rn}g^{pq}_{,n}S_{,p}S_{,q} \\ &= \frac{1}{2}\frac{dx_s}{du}\frac{dx_t}{du}(g^{rp}g_{st,p} - g^{rp}g_{ps,t} - g^{rp}g_{pt,s}).\end{aligned} \tag{5.9}$$

Thus we obtain for light rays the familiar equations

$$\frac{d^2x_r}{du^2} + \begin{Bmatrix} r & \\ m & n \end{Bmatrix}\frac{dx_m}{du}\frac{dx_n}{du} = 0, g_{mn}\frac{dx_m}{du}\frac{dx_n}{du} = 0, \tag{5.10}$$

where $\begin{Bmatrix} r & \\ m & n \end{Bmatrix}$ is the usual Christoffel symbol of the second kind.

3.3.6 3. 6. Light rays in the solar field

We note that the differential equations (5.10) for a light ray agree formally with those used in the General Theory of Relativity. Further, we have seen that the Whitehead fundamental form for the field of a massive particle can be transformed into the Schwarzschild form. It is clear then that the usual formula will be obtained for the bending of a light ray in the field of a massive particle, except for the reinterpretation of constants, of no physical interest on account of the smallness of the effect.

We can however open up new ground by investigating the behavior of light rays in the field of a *finite sphere*, for which we obtained the form (4.21) of Lecture II.

For this it is more convenient to use Lagrangian equations equivalent to (5.10). The Lagrangian is

$$2L = Ar'^2 + Br^2(\theta'^2 + \sin\theta^2\phi'^2) - 2Ccr't' - (1 - km/r)c^2t'^2 \tag{6.1}$$

where the prime indicates $\frac{d}{d\lambda}$ (λ being a parameter replacing the u of (5.10)) and A, B, C are functions of r as in (4.23) of Lecture II.

We know that we may put $\theta = \frac{1}{2}\pi$, and we have the first integrals

$$\begin{aligned} \partial L/\partial\phi' = Br^2\phi' = \alpha, \\ \partial L/\partial t' = -Ccr' - (1 - mk/r)c^2t' = -\beta c, \end{aligned} \tag{6.2}$$

where α and β are constants. Also, as in the last of (5.10), we have $L = 0$, or

$$Ar'^2 + Br^2\phi'^2 - 2Ccr't' - (1 - km/r)c^2t'^2 = 0. \tag{6.3}$$

Then (6.2) gives

$$\{A(1 - km/r) + C^2\}r'^2 + Br^2\phi'^2(1 - mk/r) = \beta^2, \tag{6.4}$$

or

$$\{A(1 - km/r) + C^2\}r'^2 + Br^2\phi'^2(1 - km/r) = (\beta^2/\alpha^2)B^2r^4\phi'^2, \tag{6.5}$$

so that

$$\{A(1 - km/r) + C^2\}(dr/d\phi)^2 + Br^2(1 - km/r) = (\beta^2/\alpha^2)B^2r^4, \tag{6.6}$$

or, with $u = 1/r$,

$$\{A(1 - kmu) + C^2\}(du/d\phi)^2 + Bu^2(1 - kmu) = (\beta^2/\alpha^2)B^2. \tag{6.7}$$

Thus we have the equation which gives the form of the ray in space:

$$\begin{aligned} (du/d\phi)^2 = F(u), \\ F(u) = \frac{(\beta^2/\alpha^2)B^2 - Bu^2(1 - kmu)}{A(1 - kmu) + C^2} \end{aligned} \tag{6.8}$$

To investigate the bending of a ray passing the sun, we put $u = u_1$ at the point of closest approach; then $F(u_1) = 0$, so that

$$(\beta^2/\alpha^2)B_1^2 - B_1u_1^2(1 - kmu_1) = 0 \tag{6.9}$$

where B_1 is the value of B for $u = u_1$. Thus we have

$$\beta^2/\alpha^2 = B_1^{-1}u_1^2(1 - kmu_1). \tag{6.10}$$

Let us approximate on the basis of small k, omitting terms of order k^2. Then C^2 is to be dropped from (6.8), and we have

$$F(u) = A^{-1}(1 - kmu)^{-1}\{B^2B^{-1}u_1^2(1 - kmu_1) - Bu^2(1 - kmu)\}. \tag{6.11}$$

Now by (4.23) of Lecture II we have approximately

$$\begin{aligned} A = 1 + kmu(1 - 2/5a^2u^2), B = 1 + 1/5kma^2u^5, \\ A(1 - kmu) = 1 - 2/5kma^2u^3 = B^{-2}. \end{aligned} \tag{6.12}$$

Thus

$$F(u) = B^4\{B_1^{-1}u_1^2(1 - kmu_1) - B^{-1}u^2(1 - kmu)\} \tag{6.13}$$

$$= B^4\{u_1^2 - u^2 - km(u_1^3 - u^3) - \frac{1}{5}kma^2(u_1^5 - u^5)\}$$

$$= u_1^2 - u^2 + \frac{4}{5}kma^2u^3(u_1^2 - u^2) - km(u_1^3 - u^3) - \frac{1}{5}kma^2(u_1^5 - u^5).$$

Writing $G_n(u) = (u_1^n - u^n)/(u_1^2 - u^2)$, we have then

$$[F(u)]^{-\frac{1}{2}} = (u_1^2 - u^2)^{-\frac{1}{2}}\{1 - \frac{2}{5}kma^2u^3 + \frac{1}{3}kmG_3 + \frac{1}{10}kma^2G_5\}. \tag{6.14}$$

In passing the sun (with shortest distance u_1^{-1} from the sun's center), a light ray is deviated through an angle

$$\gamma = -\pi + 2\int_{u=0}^{u=u_1} d\phi = -\pi + 2\int_0^{u_1} [F(u)]^{-1/2}du. \tag{6.15}$$

Now

$$\int_0^{u_1} (u_1^2 - u^2)^{-1/2}du = \frac{1}{2}\pi, \tag{6.16}$$

$$\int_o^{u_1} (u_1^2 - u^2)^{-1/2}u^3du = \frac{2}{3}u_1^3,$$

$$\int_0^{u_1} (u_1^2 - u^2)^{-1/2}G_3(u)du = 2u_1,$$

$$\int_0^{u_1} (u_1^2 - u^2)^{-1/2}G_5(u)du = \frac{8}{3}u_1^3,$$

and so the deviation is

$$\gamma = -\pi + \pi - \frac{8}{15}kma^2u_1^3 + 2kmu_1 + \frac{8}{15}kma^2u_1^3, \tag{6.17}$$

these terms representing the contributions from the separate terms of (6.14). The sun's radius a cancels out, and we arrive at the Einstein formula for the deviation of a light ray:

$$\gamma = 2kmu_1 = \frac{4Gm}{c^2r_1}, \tag{6.18}$$

where r_1, is the shortest distance from the sun's center.

3.7. Red shift in a gravitational field The Einstein prediction of a shift toward the red in the spectrum of an atom radiating in a gravitational field is based on the assumption that the ds of the General Theory of Relativity measures proper time for the atom, in the sense that the number of vibrationas of a certain spectral type occurring in an interval ds is a universal constant, independent of the situation of the radiating atom. Suppose we make the same assumption in the Whitehead theory viz. that measures proper time in this sense. Then, for an atom at rest in the field of a uniform sphere, we have by (4.21) of Lecture II,

$$ds^2 = (1 - km/r)c^2 dt^2, \tag{7.1}$$

where r is the distance of the atom from the center of the sphere and m is the mass of the sphere. We note that the radius of the sphere does not appear.

On the other hand, for the Schwarzschild form we have; as in (4.13) of Lecture I,

$$ds^2 = (1 - km/r)c^2 d^2 \tag{7.2}$$

By (4.9) of Lecture I we have dt, and so (7.1) and (7.2) agree. Thus if we take the view expressed above, the red shift is the same in Whitehead's theory as in the General Theory of Relativity.

Actually, Whitehead used a simple model of an atom (Principle of Relativity, Chap. XIII) and obtained a slightly different result. And, again using a model, he worked out the limb effect. But it would seem that this, being a question of the frequency of radiation emitted by an atom, cannot be effectively handled without a formulation in terms of quantum mechanics.

3.3.7 3.8. Wave velocity and ray velocity of light

Since, for any Galileian observer, there exists a well defined spatial background, we can in Whitehead's theory speak of wave velocity and ray velocity of light much more definitely than we can in the General Theory of Relativity, where the splitting of space-time into space-like sections is a very arbitrary procedure.

Consider the history of a phase wave or shock wave with equation $S = 0$, S being a function of the space-time coordinates. If we solve for t, this history may be written

$$S = t - \phi(x) = 0, \tag{8.1}$$

where x here stands for the three spatial coordinates. This wave advances in space in the sense of ϕ increasing, and the unit normal in the direction of this advance is

$$n_\rho = \phi_{,\rho}(\phi_{,\mu}\phi_{,\mu})^{-1/2}, \tag{8.2}$$

the comma denoting partial differentiation. If w is the wave velocity, an infinitesimal step dx_ρ following the wave for a time dt is

$$dx_\rho = wn_\rho dt. \tag{8.3}$$

But by (8.1) we have

$$dt = \phi_{,\rho}dx_\rho = w\phi_{,\rho}n_\rho dt = w(\phi_{,\rho}\phi_{,\rho})^{1/2}dt, \tag{8.4}$$

and so the wave velocity is

$$w = (\phi_{,\rho}\phi_{,\rho})^{-1/2}. \tag{8.5}$$

The components of normal slowness, in the sense of Hamilton, are

$$\sigma_\rho = n_\rho / w = \phi_{,\rho}. \tag{8.6}$$

Now we have by (8.1)

$$S_{,\rho} = -\phi_{,\rho}, S_{,4} = (ic)^{-1}, \tag{8.7}$$

and we may substitute these values in (4.20), viz. $g^{mn}S_{,m}S_{,n} = 0$, to obtain

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2(ic)^{-1}g^{\mu 4}\phi_{,\mu} - c^{-2}g^{44} = 0. \tag{8.8}$$

Substitution from (8.6) gives *Hamilton's equation*, satisfied by the components of normal slowness:

$$\Omega(x, \sigma) = g^{\mu\nu}\sigma_\mu\sigma_\nu + 2ic^{-1}g^{\mu 4}\sigma_\mu - c^{-2}g^{44} = 0. \tag{8.9}$$

This is the equation of the reciprocal wave surface at each point of space-time.

The wave velocity w in the spatial direction with direction cosines n_ρ must then satisfy, by (8.6),

$$c^{-2}g^{44}w^2 - 2ic^{-1}g^{\mu 4}n_\mu w - g^{\mu\nu}n_\mu n_\nu = 0, \tag{8.10}$$

a quadratic equation with the solutions

$$w/c = (g^{44})^{-1}[g^{\mu 4}n_\mu \pm \{-(g^{\mu 4}n_\mu)^2 + g^{44}g^{\mu\nu}n_\mu n\nu\}^{1/2}]. \tag{8.11}$$

This equation appears to give two values of w corresponding to each given normal direction n_ρ. But we must reject as extraneous any negative root, w being by definition positive by (8.5).

The ray velocity is also easily found, for by (5.5) we have, for a space-time displacement along a light ray,

$$g_{\mu\nu}dx_\mu dx_\nu + 2ig_{\mu 4}dx_\mu cdt - g_{44}c^2dt^2 = 0. \tag{8.12}$$

We put $dx_\rho = v1_\rho dt$, v being the ray velocity and 1_ρ the direction cosines of the ray. Then v satisfies the quadratic equation

$$c^2g_{44}v^{-2} - 2ig_{\mu 4}1_\mu cv^{-1} - g_{\mu\nu}1_\mu 1_\nu = 0, \tag{8.13}$$

and so

$$c/v = (g_{44})^{-1}[ig_{\mu 4}1_\mu \pm \{-(g_{\mu 4}1_\mu)^2 + g_{44}g_{\mu\nu}1_\mu 1_\nu\}^{1/2}]. \tag{8.14}$$

Here also a negative value is to be rejected as extraneous.

The formulae (8.11) and (8.14) give wave velocity and ray velocity for light in any gravitational field. let us consider now the gravitational field of a massive particle at rest, so that, as in (4.4) of Lecture I, we have

$$g_{\mu\nu} = \delta_{\mu\nu} + kmr^{-3}x_\mu x_\nu, \quad g_{\mu 4} = ikmr^{-2}x_\mu, \qquad g_{44} = 1 - km/r. \tag{8.15}$$

it is easy to verify that the conjugate tensor has the following simple form:

$$g^{\mu\nu} = \delta_{\mu\nu} - kmr^{-3}x_\mu x_\nu \quad g^{\mu 4} = -ikmr^{-2}x_\mu, \qquad (8.16)$$
$$g^{44} = 1 + km/r.$$

If ψ_w is the angle between the radius vector x_ρ (drawn from the massive particle to the point of observation) and the direction n_ρ normal to a wave, then $x_\rho n_\rho = r\cos\psi_w$ and (8.11) combined with (8.16) gives

$$w/c = (1 + km/r)^{-1}[kmr^{-1}\cos\psi_w \pm \{1 + kmr^{-1}\sin\psi_w{}^2\}^{1/2}], \qquad (8.17)$$

and if ψ_v is the angle between x_ρ and the direction 1_ρ of a wave, we have by (8.14) and (8.15)

$$c/v = (1 - km/r)^{-1}[-kmr^{-1}\cos\psi_v \pm \{1 - kmr^{-1}\sin\psi_v{}^2\}^{1/2}], \qquad (8.18)$$

or

$$v/c = (1 + km/r)^{-1}[kmr^{-1}\cos\psi_v \pm \{1 - kmr^{-1}\sin\psi_v{}^2\}^{1/2}]. \qquad (8,19)$$

For propagation in a radial direction, the ray direction coincides with the wave nrmal; sinψ_w = sin$\psi_v = 0$, and ((8.17)) is the same as (8.19) . For **outward** propagation we have $\psi_w = \psi_v = 0$ and so, if $r\rangle km$, the **single** value

$$\frac{w}{c} = \frac{v}{c} = 1. \qquad (9)$$

Whereas, if $r\langle km$, the **double** value

$$\frac{w}{c} = \frac{v}{c} = -1, or \quad \frac{km/r - 1}{km + 1} \qquad (10)$$

For,inward propagation,we have ψ_w=ψ_ν=π and so the **single**value

$$\frac{w}{c} = \frac{v}{c} = \frac{1 - km/r}{km + 1} \qquad (11)$$

if r >km,
kbut n**o value at all,** if r <km!

Thus the Schwarschild singularity, r = km, shows up as a curious point for the propagation of light!

3.4 Appendix A

3.5 Limb Effect

(**NOTE**. For many years, I assumed that the paper below had appeared in the Proceedings of the International Conference on Relativity and Gravitation in the USSR which R. M. Erdahl and I attended in 1968 and where it was delivered and accepted. Only in 2003, when my old interest in Whitehead's theory was reviving, did Prof. V. I. Yukalov inform me that the Proceedings of the Conference were never published. AJC).

WHITEHEAD' S PERTURBATION OF ATOMIC ENERGY LEVELS

A. J. Coleman
Department of Mathematics, Queen's University,
Kingston, Ontario, Canada.

Whitehead's theory of relativity implies that there is an interaction between the gravitational and electromagnetic fields such that for an atom at the surface of a star, the Coulomb potential r^{-1}between two charges must be replaced by

$$\frac{1}{r}(1 - \alpha cos^2\theta). \tag{(1)}$$

Here, θ is the angle between the radius vector joining the two interacting charges and the direction of the stellar radius passing through the $a$tom; α is a small constant depending on the strength of the gravitational field. At the surface of the sun, $\alpha = 2.12$x 10^{-6} approximately.

The effect of (1) is to perturb the normal energy levels by the small term

$$-\alpha\frac{\cos^2\theta}{r} \tag{(2)}$$

which has axial symmetry about the stellar radius through the atom. An effect of precisely this symmetry is what is needed to explain the limb-effect in the solar spectrum. One might also hope that this perturbation could account for the striking differences which have been observed in shifts within the same solar multiplet.

The effect of the perturbation (2) acting between all pairs of charge is to add

$$V^{\text{`}} = \Sigma_i\alpha\frac{Ze^2}{r}\cos^2\theta_i - \Sigma_{i<l}\frac{\alpha e^2\cos^2\theta_{ij}}{r_{ij}} \tag{(3)}$$

to the potential in Schroedinger' s equation. Here, Ze is the charge of the nucleus; 1<i, j<N , where N is the number of electrons in the atom; θ_i is the angle between $\mathbf{r}_i$ and the"vertical" ; θ_{ij} is the angle between $\mathbf{r}_{ij}$ and the vertical.

By first-order perturbation theory, the shift in energy of a J,M level is

$$\Delta E_{JM} = \langle JM|V^{'}|JM\rangle \tag{(4)}$$

For a Term with total orbital momentum L and spin S ,

$$|JM\rangle = \Sigma_{\mu+\nu=M}\langle L\mu S\nu|JM\rangle \varphi_{L\mu}U_{S\nu} \tag{(5)}$$

where $\langle$Lμ Sν|JM$\rangle$ is the vector coupling coefficient, and $\varphi_{L\mu}$ and $U_{S\nu}$ are, respectively, the appropriate pure orbital and pure spin functions. Since

$$ccs^2\vartheta = \frac{1}{3} + \frac{1}{3}(3\cos^2\vartheta - 1) \tag{(6)}$$

by using the indistinguishability of the electrons, the perturbation (4) can be expressed in the form

$$\Delta E_{JM} =< JM||V_0|JM > + < JM|V_2|JM >, \tag{(7)}$$

where

$$V_0 = \frac{\alpha}{3}N\left(\frac{Ze^2}{r_1} - \frac{N-1}{r_{12}}e^2\right), \quad and \tag{(8)}$$

$$V_2 = \frac{\alpha}{3}Ne^2\left[\frac{Z}{r_1}\left(3\cos^2\vartheta_1 - 1\right) - \frac{N-1}{2r_{12}}\left(3\cos^2\vartheta_{12} - 1\right)\right] \tag{9}$$

The advantage of this decomposition is that with respect to simultaneous rotation of all electrons about the nucleus, V_0 and V_2 belong to D_0 and D_2 representation of the rotation group, respectively.

An application of the Wigner-Eckhart theorem leads to the conclusion that

$$\Delta E_{JM} = A_L + \frac{3M^2 - J\,(J-1)}{J\,(2J-1)}B_J \tag{(10)}$$

where

$$A_L = \langle\varphi_{LL}|V_0|\varphi_{LL}\rangle, \quad B_J = \langle JJ|V_2|JJ\rangle. \tag{(11)}$$

By employing (5) and the theory of vector-coupling coefficients, a rather tedious calculation results in the formula

$$B_J = \Sigma_{\mu+\nu=J}\left|\langle L\mu S\nu|JJ\rangle\right|^2 \frac{3\mu^2 - L\,(L+1)}{L\,(2L-1)}B_L = \tag{(12)}$$

$$\{1 + \frac{3(J-L-S)(J+S-L+1)[(J+L-S+1)(J+L+S+2).-2J-3]}{L(2L-1)(2J+2)(2J+3)}\}B_L$$

where

$$B_L = \langle \varphi_{LL} | V_2 | \varphi_{LL} \rangle. \qquad ((13))$$

It follows from the Virial Theorem that

$$A_L = -2\alpha E_L \qquad ((14))$$

where $E_{:L}$ is the total energy of the state φ_{LL} which is given with sufficient accuracy for the present purposes by the mean observed energy of the Term. Thus for a Fraunhofer line, the V_0 tern gives rise to a red-shift which is proportional to the wavelength of the line and equal to 2/3 of that predic*t*ed by Einstein.

We have thus reduced the problem of calculating the Whitehead shift in the levels of a Term to that of evaluating the one constant B_L. Consequently, the shifts in the lines of a multiplet depend on two constants B^i , B^f associated with initial and final levels.

Dr. R M. Erdahl has suggested that in attempting to check this theory against observations we should treat B^i and B^f as phenomenological constants. In certain cases $B_J = 0$, so that for these the predictions are particularly simple. For example, from (13) it follows immediately that $B_L = 0$ if L = 0, that is for *an* S-term. But it follows from (12) that B_J also vanishes for states such as $^4P_{1/2}$, $^6D_{1'2}$, ... $^{10}F_{11/2}$ and many others. It may also be worth looking at Terns for which B_J is small.

To test the usefulness of Whitehead's perturbation in explaining the actual complex observations of shifts in the solar s*p*ectrum, it would be particularly valuable to have reliable measurements for the absolute shifts at various points in the solar disc for all lines of a multiplet and especially for multiplets which include one or more transitions between energy levels with symmetry type appearing in the list described above.

In addition to possible perturbation of energy levels by a gravo-electric interaction, the Fraunhofer lines are undoubtedly shifted by Doppler and pressure effects. To this must be added the classic Einstein shift which has been confirmed by the Pound-Rebka experiment and which follows from Newton's theory and the conservation of energy. The Einstein and Doppler shifts are proportional to the wave-length of the line and by themselves certainly cannot explain the observed shifts in the solar spectrum.

If Whitehead's perturbation combined with reasonable assumptions about pressure shift is unable to explain the observations, all is not lost. If the astronomers can obtain reliable observations, especially at the limb, of a large number of multiplets of diverse symmetry, it should be possible, using the techniques of the present paper, to obtain a good approximation for a perturbation of atomic energy levels which would explain the observations by employing a multipole analysi*s*.

Since in the solar spectrum, the observed deviations from Einstein's predicted shift are as large or larger than his prediction, it is clearly of great interest to establish the source of this deviation in order to be able to interpret spectral shifts from other stars with any confidence.

August 16, 1968.

4 Appendix B: Figures

Figure I.1

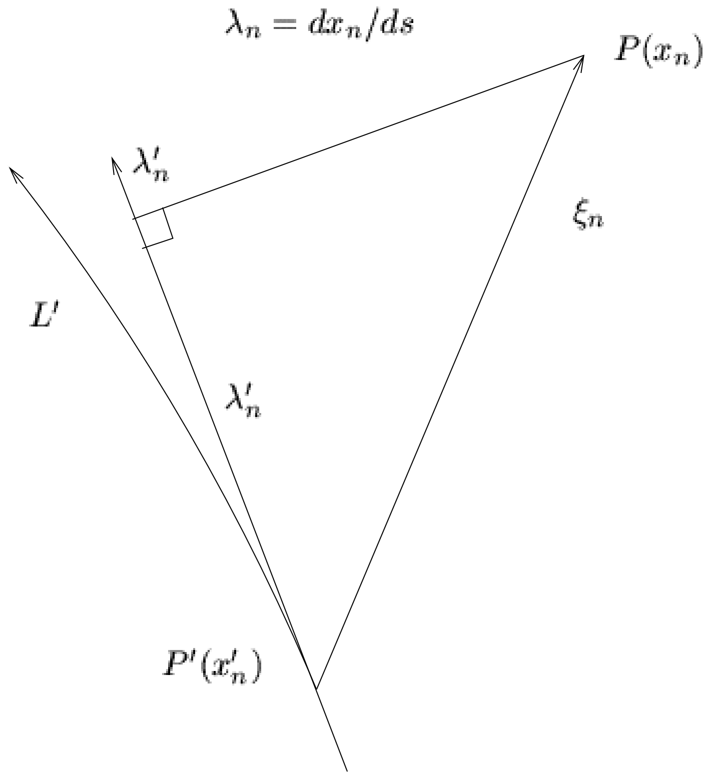

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Figure I.2

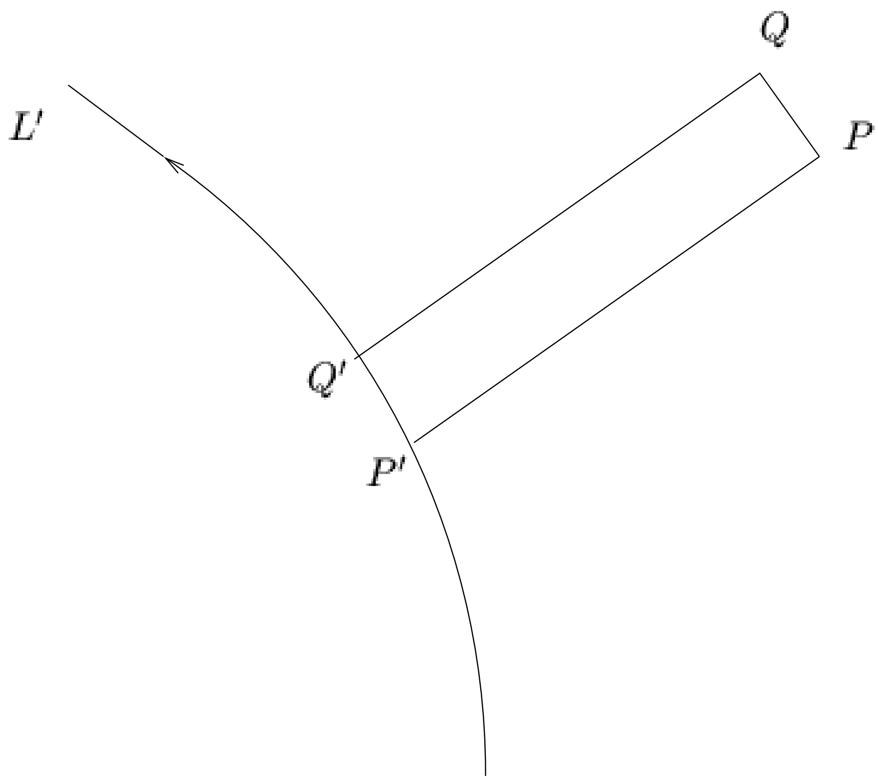

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Figure I.3

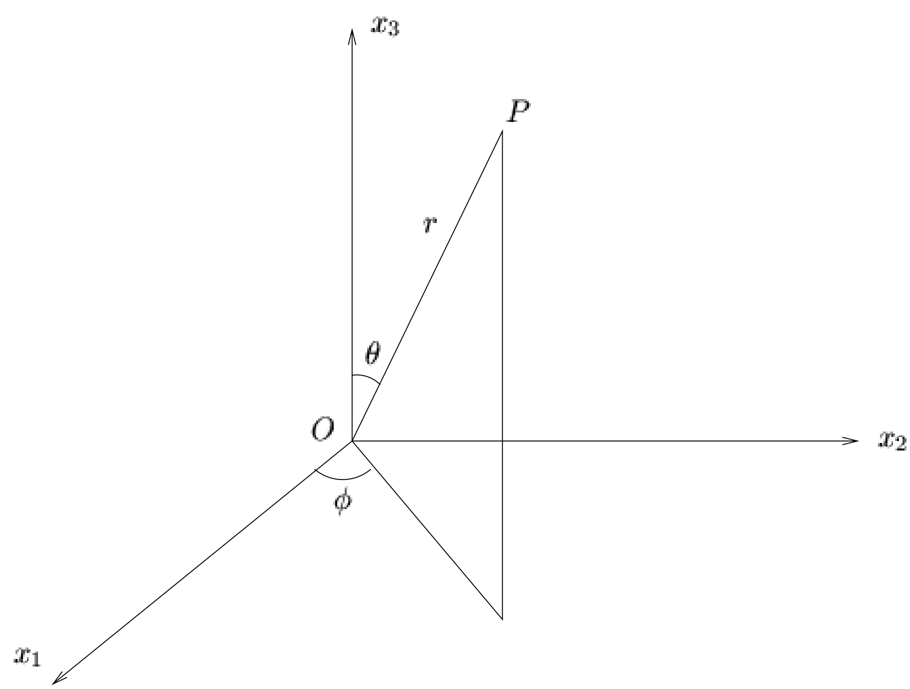

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Figure II.1

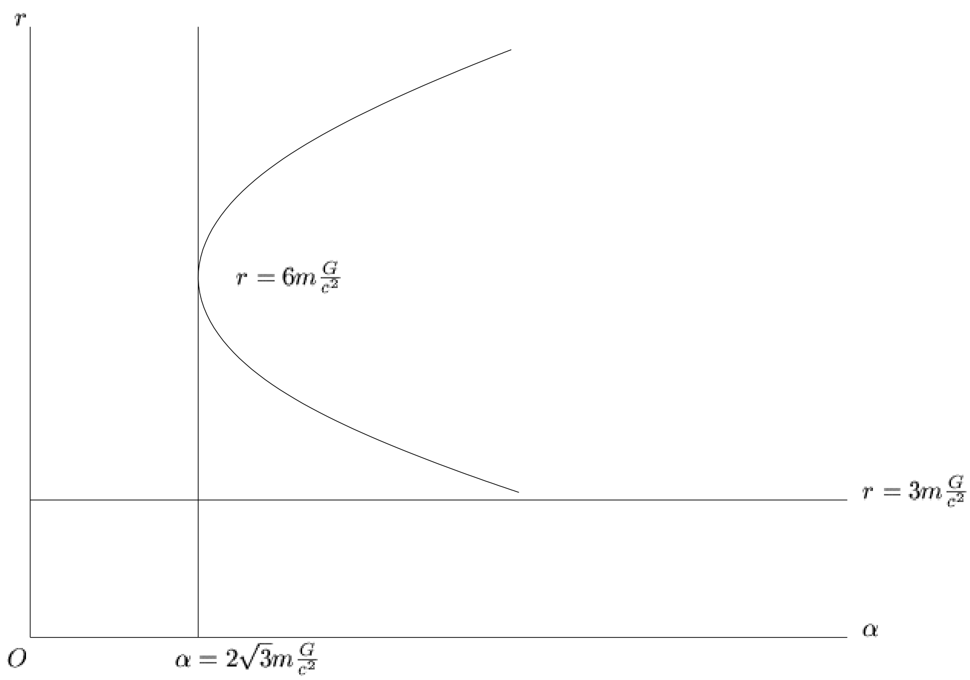

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Figure II.2

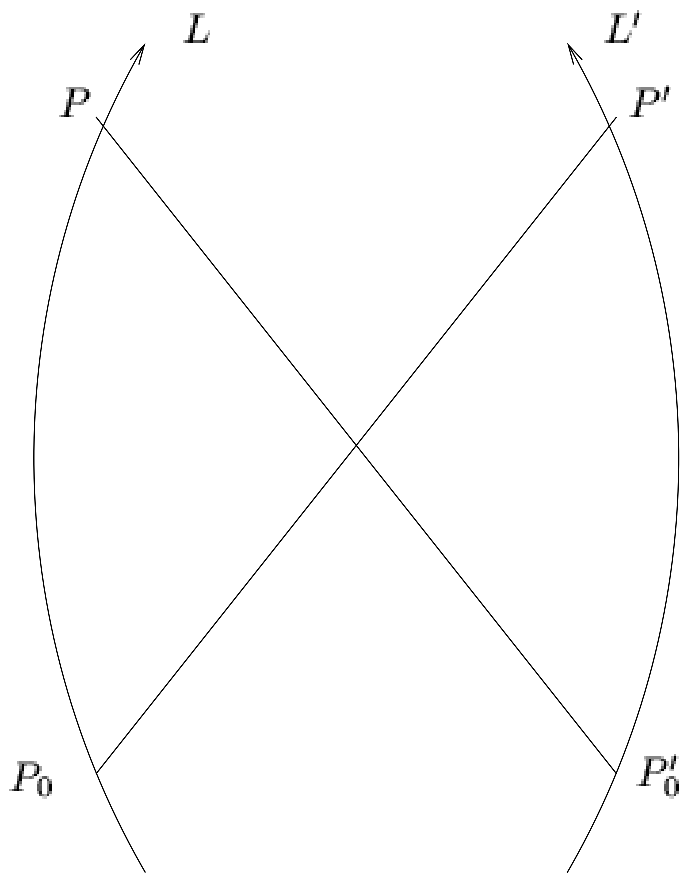

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Figure III.1